MATH534A, Problem Set 7, due October 30

All problems are worth the same number of points.

1. Let $F_i\colon M \to N_i$, where $i \in \{1, \dots, k\}$, be smooth maps, and assume that one of them is an immersion. Show that the map $F\colon M \to N_1 \times \cdots \times N_k$, given by $F(x) = (F_1(x), \dots, F_k(x))$, is an immersion.

2. (a) Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be a map whose components are given by

$$f_i = \begin{cases} x_i \sqrt{1 + \sum_{i=k+1}^{n} x_i^2} & \text{for } i \le k, \\ x_i & \text{for } i > k, \end{cases}$$

   where $k \in \{1, \dots, n\}$ is given. Show that F is a diffeomorphism.

   (b) Show that the restriction of F to a subset of $\mathbb{R}^n$ given by $\sum_{i=1}^{k} x_i^2 = 1$ is an embedding.

   (c) Hence show that the subset of $\mathbb{R}^n$ given by

$$\sum_{i=1}^{k} x_i^2 \; - \sum_{i=k+1}^{n} x_i^2 = 1$$

   is a submanifold diffeomorphic to $S^{k-1} \times \mathbb{R}^{n-k}$. (By definiton, S^0 is a two-point set with discrete topology.)

3. Let M be a compact manifold, and let $C_1, C_2 \subset M$ be its closed disjoint subsets. Let also f_1, f_2 be smooth functions defined on some open neighborhoods of C_1, C_2 respectively. Show that there exists a smooth function $f\colon M \to \mathbb{R}$ such that $f|_{C_1} = f_1$, $f|_{C_2} = f_2$.

4. Let M be a compact manifold, N be its compact submanifold, and let $f\colon N \to \mathbb{R}$ be a smooth function. Show that there exists a smooth function $\tilde{f}\colon M \to \mathbb{R}$ such that $\tilde{f}|_N = f$.

5. Prove that any open cover of a second countable topological space admits a countable subcover.