MATH534A, Problem Set 7, solutions

Problem 1. Let $F_i\colon M \to N_i$, where $i \in \{1,\dots,k\}$, be smooth maps, and assume that one of them is an immersion. Show that the map $F\colon M \to N_1 \times \cdots \times N_k$, given by $F(x) = (F_1(x),\dots,F_k(x))$, is an immersion.

Solution. Without loss of generality assume that $F_1\colon M \to N_1$ is an immersion. Let also $\pi_1\colon N_1 \times \cdots \times N_k \to N_1$ be the projection to the first factor. It follows from the construction of the smooth structure on $N_1 \times \cdots \times N_k$ that the map π_1 is smooth. Furthermore, we have $\pi_1 \circ F = F_1$, so

$$d_{F(p)}\pi_1 \circ d_pF = d_pF_1 \tag{1}$$

for any point $p \in M$. Now, assume that F is not an immersion. Then there is $p \in M$ and $v \in T_pM$, $v \neq 0$ such that $d_pF(\xi) = 0$. Then, applying both sides of (1) to v, we get $d_pF_1(\xi) = 0$, which contradicts the assumption that F_1 is an immersion. So, F is an immersion.

Problem 2.

1. Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be a map whose components are given by

$$f_i = \begin{cases} x_i\sqrt{1+\sum\limits_{i=k+1}^{n} x_i^2} & \text{for } i \leq k, \\ x_i & \text{for } i > k, \end{cases}$$

where $k \in \{1,\dots,n\}$ is given. Show that F is a diffeomorphism.

2. Show that the restriction of F to a subset of $\mathbb{R}^n$ given by $\sum_{i=1}^{k} x_i^2 = 1$ is an embedding.

3. Hence show that the subset of $\mathbb{R}^n$ given by

$$\sum_{i=1}^{k} x_i^2 - \sum_{j=k+1}^{n} x_j^2 = 1 \tag{2}$$

is a submanifold diffeomorphic to $S^{k-1} \times \mathbb{R}^{n-k}$. (By definiton, S^0 is a two-point set with discrete topology.)

Solution. 1. The map F is smooth, because its components are smooth. Furthermore, F has an inverse whose components are

$$g_i = \begin{cases} \dfrac{x_i}{\sqrt{1+\sum_{j=k+1}^{n} x_j^2}} & \text{for } i \leq k, \\ x_i & \text{for } i > k, \end{cases}$$

so F^{-1} is smooth as well, meaning that F is a diffeomorphism.

2. F is a diffeomorphism, and hence an embedding. Therefore, the restriction of F to any submanifold is an embedding as well.

3. One can find the image of the submanifold $S = \{\sum_{i=1}^{k} x_i^2 = 1\}$ under F by plugging in the components of F^{-1} for x_i's. This gives

$$\sum_{i=1}^{k} \frac{x_i^2}{1 + \sum_{j=k+1}^{n} x_j^2} = 1,$$

which is equivalent to (2). So, (2) defines a submanifold diffeomorphic to S. To complete the proof, we represent $\mathbb{R}^n$ as $\mathbb{R}^k \times \mathbb{R}^{n-k}$, where a point $(x_1, \dots, x_n) \in \mathbb{R}^n$ is identified with the pair $((x_1, \dots, x_k), (x_{k+1}, \dots, x_n)) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$. In this representation, the submanifold $S \subset \mathbb{R}^n$ consists of pairs of the form (a, b), where $a \in \mathbb{R}^k$ belongs to the unit sphere S^{k-1}, and $b \in \mathbb{R}^{n-k}$ is arbitrary. So, S is diffeomorphic to a direct product $S^{k-1} \times \mathbb{R}^{n-k}$, and so is the manifold defined by (2), as desired.

Problem 3. Let M be a compact manifold, and let $C_1, C_2 \subset M$ be its closed disjoint subsets. Let also f_1, f_2 be smooth functions defined on some open neighborhoods of C_1, C_2 respectively. Show that there exists a smooth function $f \colon M \to \mathbb{R}$ such that $f|_{C_1} = f_1$, $f|_{C_2} = f_2$.

Solution. Let U_1, U_2 be the domains of f_1 and f_2 respectively. Define also $U_3 = M \setminus (C_1 \cup C_2)$. Then U_1, U_2, U_3 form an open cover of M. Let ϕ_1, ϕ_2, ϕ_3 be a partition of unity subordinate to that cover. Define a function f by the rule

$$f = \phi_1 f_1 + \phi_2 f_2, \tag{3}$$

where $\phi_i f_i$ is understood as 0 whenever $\phi_i = 0$. To prove that f is smooth, it suffices to show that $\phi_1 f_1$, $\phi_2 f_2$ are both smooth. We prove that $f_1 \phi_1$ is smooth, while the proof of smoothness for $\phi_2 f_2$ is analogous. (Smoothness of such a function was in fact proved in class, but we also give the proof here for the sake of completeness.) To prove smoothness of $\phi_1 f_1$, note that f_1 and ϕ_1 are both smooth in U_1, and hence so is their product. So it suffices to prove smoothness of $\phi_1 f_1$ at each point $p \notin U_1$. For such points, we have $p \notin \operatorname{supp} \phi_1$ (since $\operatorname{supp} \phi_1 \subset U_1$ by definition of partition of unity subordinate to a cover), so $p \in M \setminus \operatorname{supp} \phi_1$. The latter set is open (since $\operatorname{supp} \phi_1$ is, by definition, closed) and for any its point we have $\phi_1 f_1 = 0$ (since $\phi_1 = 0$ outside $\operatorname{supp} \phi_1$ by definition of support). So, there is an open neighborhood of p where $\phi_1 f_1 = 0$, which means that $\phi_1 f_1$ is smooth at p, as desired.

Addendum. The above solution is not perfectly correct, because function (3) in general does not satisfy $f|_{C_1} = f_1$ and $f|_{C_2} = f_2$. For example, assume that $U_1 = U_2 = M$. In this case, as a partition of unity subordinate to the cover U_1, U_2, U_3, we can take, for instance, $\phi_1 = 1, \phi_2 = 0, \phi_3 = 0$. Then $f = f_1$, which is not equal to f_2 on C_2.

For (3) to satisfy $f|_{C_1} = f_1$ and $f|_{C_2} = f_2$, we need to have $\phi_1|_{C_2} = 0$ (which also implies $\phi_1|_{C_1} = 1$) and $\phi_2|_{C_1} = 0$ (which also implies $\phi_2|_{C_2} = 1$). This is true provided that

$$U_1 \cap C_2 = \emptyset, \quad U_2 \cap C_1 = \emptyset. \tag{4}$$

Although (4) might not be true for initially given U_1, U_2 (and thus the above solution is, in general, not correct), it can be always arranged by shrinking U_1, U_2. Indeed, if the initially given U_1 intersects C_2, then we can replace it with a smaller set $U_1 \setminus C_2$, which is still an open neighborhood of C_1. Similarly, we can replace U_2 with $U_2 \setminus C_1$.

So, the above solution becomes correct if we assume (without loss of generality) that $U_1 \cap C_2 = \emptyset$ and $U_2 \cap C_1 = \emptyset$.

Problem 4. Let M be a compact manifold, N be its compact submanifold, and let $f \colon N \to \mathbb{R}$ be a smooth function. Show that there exists a smooth function $\tilde{f} \colon M \to \mathbb{R}$ such that $\tilde{f}|_N = f$.

Solution. By definition of a submanifold, for any $p \in N$ there is a chart U_p in M containing p such that the intersection $U_p \cap N$ is given by linear equations in terms of the coordinates in U_p.

Taking such charts for every p, we get an open cover of N. Since N is compact, that cover admits a finite subcover, which we denote by $U_1, \ldots, U_k$. In each U_i, the submanifold N is given by equations $x_{n+1} = \cdots = x_m = 0$ (where $m = \dim M$, $n = \dim N$), while $x_1, \ldots, x_n$ form a local coordinate system on N. So, $f|_{U_i \cap N}$ can be written as $f = f(x_1, \ldots, x_n)$, which allows one to extend f to U_i by setting

$$\tilde{f}_i(x_1, \ldots, x_m) = f(x_1, \ldots, x_n).$$

In this way, we get a bunch of functions $\tilde{f}_i \colon U_i \to \mathbb{R}$ such that

$$\tilde{f}_i|_{U_i \cap N} = f.$$

Now, take $U_{k+1} = M \setminus N$, and let $\phi_1, \ldots, \phi_{k+1}$ be a partition of unity subordinate to the cover $U_1, \ldots, U_{k+1}$. Then one can define the desired extension $\tilde{f}$ by the formula

$$\tilde{f} = \sum_{i=1}^{k} \phi_i f_i.$$

This function is smooth on M, as follows from the argument we used in Problem 3. Furthermore, for any $p \in N$ we either have $f_i(p) = f(p)$, or $\phi_i(p) = 0$. In both cases, one has $\phi_i(p) f_i(p) = \phi_i(p) f(p)$, so

$$\tilde{f}(p) = \sum_{i=1}^{k} \phi_i(p) f_i(p) = \sum_{i=1}^{k} \phi_i(p) f(p) = f(p) \sum_{i=1}^{k} \phi_i(p) = f(p),$$

as desired.

Problem 5. Prove that any open cover of a second countable topological space admits a countable subcover.

Solution. Let X be a second countable topological space, $\mathbb{B}$ be its countable base, and let $\mathbb{U}$ be any cover of X. Let

$$\tilde{\mathbb{B}} = \{B \in \mathbb{B} \mid \exists\, U \in \mathbb{U} \text{ such that } B \subset U\}$$

be the set of those elements of the base that are contained in at least one element of the cover. For any $B \in \tilde{\mathbb{B}}$, let U_B be any element of the cover such that $B \subset U_B$ (such U_B exists by construction of $\tilde{\mathbb{B}}\}$. Define

$$\tilde{\mathbb{U}} = \{U_B \mid B \in \tilde{\mathbb{B}}\}.$$

This is a countable subset of $\mathbb{U}$, so it remains to show that it is a cover. Take any $x \in X$. Then there is $U \in \mathbb{U}$ such that $x \in U$ (because $\mathbb{U}$ is a cover). Furthermore, there exists $B \in \mathbb{B}$ such that $x \in B$, and $B \subset U$ (because $\mathbb{B}$ is a base). Since $B \subset U$, it follows that $B \in \tilde{\mathbb{B}}$. So, there is an element of $\tilde{\mathbb{U}}$, namely U_B, which contains B. But this implies $x \in U_B$. So, every point of X belongs to some element of $\tilde{\mathbb{U}}$, which means that $\tilde{\mathbb{U}}$ is a cover of X, as desired.