MATH534A, Problem Set 8, due Nov 13

All problems are worth the same number of points.

1. Let M, N be smooth manifolds of the same dimension, $F\colon M \to N$ be a smooth map, and let $A \subset M$ have measure 0. Prove that $F(A)$ has measure 0.

2. Let M be a smooth manifold, and let $A \subset M$ have measure 0. Prove that $A \neq M$.

3. Let M be a smooth manifold of dimension m, and let $\pi\colon TM \to M$ be the natural projection, $\pi(p, v) = p$. For a chart (U, ϕ) on M, define $\hat{\phi}\colon \pi^{-1}(U) \to \mathbb{R}^m \times \mathbb{R}^m$ by
$$\hat{\phi}(p, v) = (\phi(p), d_p\phi(v)).$$
   (a) Prove that there is a unique topology on TM such that for any chart (U, ϕ) on M the set $\pi^{-1}(U) \subset TM$ is open, and $\hat{\phi}$ maps $\pi^{-1}(U)$ homeomorphically to an open subset of $\mathbb{R}^{2m}$.
   (b) Show that TM with this topology is Hausdorff and second countable. Hence show that TM endowed with charts $(\pi^{-1}(U), \hat{\phi})$ is a smooth manifold.

4. (a) Let $F\colon M \to N$ be a smooth map. Consider the map $dF\colon TM \to TN$ given by $dF(p, v) = (F(p), d_pF(v))$. This map is sometimes called the *global differential of* F (but it is also okay to call it the differential of F). Prove that this map is smooth.
   (b) Let $F\colon M \to \mathbb{R}^n$ be a smooth map. Consider the map $TM \to \mathbb{R}^n$ given by $(p, v) \mapsto d_pF(v)$. Prove that this map is smooth.

5. (a) Let $F\colon M \to N$ be a diffeomorphism, and let v be a smooth vector field on M. Prove that F_*v is a smooth vector field on N.
   (b) Prove that this is in general not true if F is just a smooth bijection.
   Hint: Compute the pushforward of the vector field $\frac{\partial}{\partial x}$ by the map $\mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^3$.

6. One says that the tangent bundle to M is *trivial* if there exists a diffeomorphism $\xi\colon TM \to M \times \mathbb{R}^m$ (where m is the dimension of M) which is of the form $\xi(p, v) = (p, \dots)$, and moreover the restriction of ξ to T_pM is a vector space isomorphism between T_pM and $\{p\} \times \mathbb{R}^m$.
   (a) Prove that M has trivial tangent bundle if and only if M is *parallelizable*, which means that it admits smooth vector fields $v_1, \dots, v_m$ which form a basis of T_pM at every point.
   (b) Prove that $\mathbb{R}^n$ and S^1 are parallelizable manifolds.

   Remark: We will see later in the course that not all manifolds are parallelizable. In particular, S^2 and $\mathbb{RP}^2$ are not (for S^2 this follows e.g. from *hairy ball theorem*). Moreover, the tangent bundle for each of these manifolds is not diffeomorphic to the direct product of the manifold itself and $\mathbb{R}^2$. (This is stronger then non-triviality of the tangent bundle, because triviality means that there exists a diffeomorphism $TM \to M \times \mathbb{R}^m$ with additional properties.)