MATH534A, Exam 1, Soultions

Problem 2. Let S^2 be the standard unit sphere, $U \subset S^2$ be the northern hemisphere, and p be a point in U. Let also (x_1, x_2) be coordinates in U given by $x_1(x,y,z) = x$, $x_2(x,y,z) = y$, and let $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$ be the associated basis of T_pM. Compute $\mathrm{d}i(\frac{\partial}{\partial x_1})$, where $i\colon S^2 \to \mathbb{R}^3$ is the inclusion map, and $\mathrm{d}i$ is its differential (at p).

Denote by p_x, p_y, p_z the coordinates of p in $\mathbb{R}^3$.

Solution 1 (using smooth curves). By definition of the differential in terms of smooth curves, we have

$$\mathrm{d}i(\frac{\partial}{\partial x_1}) = \frac{d}{dt}\bigg|_{t=0} i(\gamma(t)),$$

where $\gamma(t)$ is any smooth curve in S^2 such that $\gamma(0) = p$, $\gamma'(0) = \frac{\partial}{\partial x_1}$. In coordinates (x_1, x_2), a curve γ in S^2 near p can be represented by two functions $x_1(t)$, $x_2(t)$. In terms of these functions, the condition $\gamma(0) = p$ becomes $x_1(0) = p_x$, $x_2(0) = p_y$. Furthermore, the coordinates of the tangent vector $\gamma'(0)$ in the basis $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$ are $x_1'(0)$, $x_2'(0)$. So, $\gamma'(0) = \frac{\partial}{\partial x_1}$ is equivalent to $x_1'(0) = 1$, $x_2'(0) = 0$. Thus, as γ we can take the curve given in coordinates (x_1, x_2) by

$$x_1(t) = p_x + t, \quad x_2(t) = p_y.$$

Further, $i(\gamma(t)) = \gamma(t)$ where in the right-hand side we interpret γ as a curve in $\mathbb{R}^3$. In $\mathbb{R}^3$, the curve γ is given by three functions

$$x(t) = p_x + t, \quad y(t) = p_y, \quad z(t) = \sqrt{1 - x(t)^2 - y(t)^2} = \sqrt{1 - (p_x + t)^2 - p_y^2}.$$

Therefore,

$$\mathrm{d}i(\frac{\partial}{\partial x_1}) = \frac{d}{dt}\bigg|_{t=0} i(\gamma(t)) = \frac{d}{dt}\bigg|_{t=0} (p_x + t, p_y, \sqrt{1 - (p_x + t)^2 - p_y^2}) = (1, 0, -\frac{p_x}{p_z}).$$

Since we represent the curve γ in $\mathbb{R}^3$ by its x, y, z coordinates, the latter expression gives coordinates of $\mathrm{d}i(\frac{\partial}{\partial x_1})$ in the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. So, another possible form of the answer is:

$$\mathrm{d}i(\frac{\partial}{\partial x_1}) = \frac{\partial}{\partial x} - \frac{p_x}{p_z}\frac{\partial}{\partial z}.$$

Note that $\frac{\partial}{\partial z}$ component can also be found using the fact that the resulting vector should be orthogonal to (p_x, p_y, p_z), and there is in fact no need to differentiate the square root function.

Solution 2 (using Jacobian matrices). Take (x_1, x_2) coordinates in S^2 and standard (x, y, z) coordinates in $\mathbb{R}^3$. In these coordinates, the mapping i can be written as

$$x = x_1, \quad y = x_2, \quad z = \sqrt{1 - x_1^2 - x_2^2}. \tag{1}$$

The differential of i is a linear mapping $T_pS^2 \to T_p\mathbb{R}^3$ whose matrix in bases $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$ for T_pS^2 and $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ for $T_p\mathbb{R}^3$ is the Jacobian of (1). Furthermore, by definition of the matrix of a linear map, the image of the first basis vector is the first column, so

$$\mathrm{d}i(\frac{\partial}{\partial x_1}) = \left.\begin{pmatrix} \frac{\partial x}{\partial x_1} \\ \frac{\partial y}{\partial x_1} \\ \frac{\partial z}{\partial x_1} \end{pmatrix}\right|_p = \left.\begin{pmatrix} 1 \\ 0 \\ -\frac{x_1}{\sqrt{1 - x_1^2 - x_2^2}} \end{pmatrix}\right|_p = \frac{\partial}{\partial x} - \frac{p_x}{p_z}\frac{\partial}{\partial z}.$$

Solution 3 (using differential operators). By definition of the differential, for any function $f = f(x, y, z)$ in $\mathbb{R}^3$, we have

$$\mathrm{d}i(\frac{\partial}{\partial x_1})f = \frac{\partial}{\partial x_1}(i^* f) = \frac{\partial}{\partial x_1}(f \circ i) = \frac{\partial}{\partial x_1}(f|_{S^2}).$$

In coordinates (x_1, x_2), we have

$$f|_{S^2} = f(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}),$$

so

$$\begin{aligned}\mathrm{d}i(\frac{\partial}{\partial x_1})f = \frac{\partial}{\partial x_1}(f|_{S^2}) &= \frac{\partial}{\partial x_1} f(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}) \\ &= \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial}{\partial x_1}\sqrt{1 - x_1^2 - x_2^2}\right)\Big|_p = (\frac{\partial}{\partial x} - \frac{p_x}{p_z}\frac{\partial}{\partial z})f,\end{aligned}$$

meaning that

$$\mathrm{d}i(\frac{\partial}{\partial x_1}) = \frac{\partial}{\partial x} - \frac{p_x}{p_z}\frac{\partial}{\partial z}.$$

Problem 3. Let S^2 be the standard unit sphere, $n \in S^2$ be its north pole, and let $\sigma \colon S^2 \setminus \{n\} \to \mathbb{C}$ be the stereographic projection from n (here we identify $\mathbb{R}^2$ and $\mathbb{C}$). Define a map $\phi \colon S^2 \to \mathbb{CP}^1$ by

$$\phi(p) = \begin{cases} [\sigma(p) : 1] & \text{if } p \neq n, \\ [1 : 0] & \text{if } p = n. \end{cases}$$

Prove that ϕ is a diffeomorphism.

Solution. Let $s \in S^2$ be the south pole. Then we have a smooth altas $\{(U, \sigma), (\tilde{U}, \tilde{\sigma})\}$ on S^2, where $U = S^2 \setminus \{n\}$, $\tilde{U} = S^2 \setminus \{s\}$, and $\tilde{\sigma} \colon \tilde{U} \to \mathbb{C}$ is the stereographic projection from s.

Similarly, we have an altas $\{(V, \xi), (\tilde{V}, \tilde{\xi})\}$ on $\mathbb{CP}^1$, where $V = \mathbb{CP}^1 \setminus \{[1 : 0]\}$, $\tilde{V} = \mathbb{CP}^1 \setminus \{[0 : 1]\}$, $\xi([x, y]) = x/y$, $\tilde{\xi}([x, y]) = y/x$.

Note that ϕ maps U bijectively to V. Indeed, $\phi(U) \subset V$ by definition of ϕ. Furthermore, $\phi|_U = i \circ \sigma$, where $i \colon \mathbb{C} \to V$ is given by $i(z) = [z : 1]$. Both i and σ are bijections, so $\phi|_U \colon U \to V$ is a bijection. And since we also have $\phi(n) = [1 : 0]$, it follows that ϕ maps $S^2 = U \sqcup \{n\}$ bijectively to $\mathbb{CP}^1 = V \sqcup \{[1 : 0]\}$.

Now we prove that ϕ is smooth. First take $p \in U$. Then $\phi(U) \in V$, so to show that ϕ is smooth at p we can use the coordinate representation of ϕ in charts (U, σ), (V, ξ). This coordinate representation is

$$\xi \circ \phi \circ \sigma^{-1}(z) = \xi([\sigma \circ \sigma^{-1}(z) : 1]) = \xi([z : 1]) = z,$$

i.e. it is the identity map and hence smooth. (In this computation we used that $\phi(p) = [\sigma(p) : 1]$ in U.) So it remains to verify smoothness of ϕ at n. To do that, it suffices to take any chart whose domain contains n and any chart whose domain contains $\phi(n) = [1 : 0]$, and then verify smoothness of the corresponding coordinate representation of ϕ. As such charts, we take $(\tilde{U}, \tilde{\sigma})$ and $(\tilde{V}, \tilde{\xi})$. The corresponding coordinate representation of ϕ is $\tilde{\xi} \circ \phi \circ \tilde{\sigma}^{-1}(z)$. For $z \neq 0$, we have $\tilde{\sigma}^{-1}(z) \neq n$, so

$$\tilde{\xi} \circ \phi \circ \tilde{\sigma}^{-1}(z) = \tilde{\xi}([\sigma \circ \tilde{\sigma}^{-1}(z) : 1]) = \tilde{\xi}([1/\bar{z} : 1]) = \bar{z}.$$

But for $z = 0$ we have $\tilde{\xi} \circ \phi \circ \tilde{\sigma}^{-1}(z) = \tilde{\xi} \circ \phi(n) = \tilde{\xi}([1 : 0]) = 0 = \bar{z}$, so the formula

$$\tilde{\xi} \circ \phi \circ \tilde{\sigma}^{-1}(z) = \bar{z}$$

is actually valid for any $z \in \mathbb{C}$, which proves that $\tilde{\xi} \circ \phi \circ \tilde{\sigma}^{-1}(z)$ is smooth, and thus ϕ is smooth at n (and hence everywhere).

These computations also show that, first, ϕ maps U diffeomorphically to V, and second, ϕ maps some open neighborhood of n diffeomorphically to an open neighborhood of $[1 : 0]$ (it is not hard to see that in fact ϕ maps $\tilde{U}$ diffeomorphically to $\tilde{V}$). Since U and an open neighborhood of n cover S^2, it follows that ϕ is a local diffeomorphism. And since it is also bijective, it is actually a global diffeomorphism, as desired.

Problem 4. Prove that the set $\{(x, y, z) \in \mathbb{RP}^2 \mid xy = z^2\}$[1] is a smooth submanifold of $\mathbb{RP}^2$.

Solution 1. First note that even though the coordinates x, y, z for a point in $\mathbb{RP}^2$ are only defined up to a common non-zero factor, the equation $xy = z^2$ is invariant under such rescaling of variables and hence the set of its solutions in $\mathbb{RP}^2$ is well-defined. Now we prove that this set is a submanifold. We have the following charts in $\mathbb{RP}^2$: $U_x = \{(x : y : z) \in \mathbb{RP}^2 \mid x \neq 0\}, U_y = \{(x : y : z) \in \mathbb{RP}^2 \mid y \neq 0\}, U_z = \{(x : y : z) \in \mathbb{RP}^2 \mid z \neq 0\}$. To prove that $S = \{(x : y : z) \in \mathbb{RP}^2 \mid xy = z^2\}$ is a submanifold of $\mathbb{RP}^2$, it suffices to show that $S \cap U_x$ is a submanifold of U_x, $S \cap U_y$ is a submanifold of U_y, and $S \cap U_z$ is a submanifold of U_z. Indeed, let $p \in S$. Then $p \in U_x$, or $p \in U_y$, or $p \in U_z$. Assume, for example, that $p \in U_x$. Then, if we know that $S \cap U_x$ is a submanifold of U_x, it follows that there is an open subset $U \ni p$ of U_x such that $S \cap U$ is, in appropriate coordinates, a vector subspace. But U is also an open subset of $\mathbb{RP}^2$, which means that S satisfies the definition of a submanifold of $\mathbb{RP}^2$ near p. Applying this argument for every $p \in \mathbb{RP}^2$, we get that S is a submanifold.

Now we show that $S \cap U_x$, $S \cap U_y$, $S \cap U_z$ are sumbanifolds of the corresponding charts. In U_x, we have coordinates $x_1 = y/x$, $x_2 = z/x$. The equation $xy = z^2$ in this chart can be rewritten as $y/x = (z/x)^2$, so $S \cap U_x = \{(x_1, x_2) \in U_x \mid x_1 = x_2^2\}$. It is a graph of a smooth function and hence a submanifold. Similarly, in U_y we have $x_1 = x/y$, $x_2 = z/y$, and the equation $xy = z^2$ is equivalent to $x_1 = x_2^2$, which is also a submanifold. Finally, in U_z we have $x_1 = x/z$, $x_2 = y/z$, and the equation $xy = z^2$ is equivalent to $x_1 x_2 = 1$, which is the graph of $x_2 = 1/x_1$ and hence a submanifold.

Also note that in fact $S \subset U_x \cup U_y$. Indeed, the only point of $\mathbb{RP}^2$ not belonging to $U_x \cup U_y$ is $(0 : 0 : 1) \notin S$. So, since $U_x \cup U_y$ cover S, we do not really need to consider U_z.

Solution 2. $xy - z^2$ is a non-degenerate indefinite quadratic form, so the set of points in $\mathbb{R}^3$ satisfying the equation $xy - z^2 = 0$ is a cone (see `https://en.wikipedia.org/wiki/Quadric#Euclidean_space`). Therefore, there exists a non-zero linear function $l(x, y, z) = ax + by + cz$ such that the plane $l(x, y, z) = 0$ intersects the cone $xy = z^2$ only its vertex $(0, 0, 0)$. For instance, one can take $l(x, y, z) = x + y$. Indeed,

$$\begin{cases} xy = z^2, \\ y = -x \end{cases} \Leftrightarrow \begin{cases} -y^2 = z^2, \\ y = -x \end{cases} \Leftrightarrow \begin{cases} \begin{cases} y = 0, \\ z = 0, \end{cases} \\ y = -x \end{cases} \Leftrightarrow \begin{cases} x = 0, \\ y = 0, \\ z = 0. \end{cases}$$

Therefore, the set $S = \{(x : y : z) \in \mathbb{RP}^2 \mid xy = z^2\}$ is completely contained in the open set $U = \{(x : y : z) \in \mathbb{RP}^2 \mid l(x, y, z) \neq 0\}$, and it suffices to show that S is a submanifold of U. To that end, consider the function on U given by

$$f([x : y : z]) = \frac{xy - z^2}{l(x, y, z)^2}.$$

It is well-defined because the numerator and denominator are of the same degree, and the denominator does not vanish. Since S is the zero set of f, it remains to show that the differential of f does not vanish at points of S. To that end, we rewrite f in local coordinates, which is achieved by dividing both the numerator and denominator by x^2, y^2, or z^2 (depending on the chart). So, the coordinate representation of f will be of the form $f = a/b$, where $a = xy - z^2$, divided by the square of one of the variables, and $b = l(x, y, z)^2$, divided by the square of the same variable. Therefore,

$$\operatorname{grad} f = \operatorname{grad} \frac{a}{b} = \frac{b \cdot \operatorname{grad} a - a \cdot \operatorname{grad} b}{b^2}.$$

At points of S, we have $a = 0$, so

$$\operatorname{grad} f = \operatorname{grad} \frac{a}{b} = \frac{1}{b} \cdot \operatorname{grad} a.$$

[1] The formula should in fact read $\{(x : y : z) \in \mathbb{RP}^2 \mid xy = z^2\}$.

Possible forms of the function a are $x_1 - x_2^2$ and $x_1 x_2 - 1$. Both have non-vanishing gradients on their zero sets, so S is a submanifold.

Solution 3. The quadratic form $xy - z^2$ has signature $(+, -, -)$, therefore there is an invertible linear transformation mapping it to the form $-x^2 - y^2 + z^2$. This linear transformation can be written as $(x, y, z) \mapsto (x, y, z)A$, where A is a suitable 3×3 matrix. It induces a mapping from $\mathbb{RP}^2$ to itself given by the formula

$$[x : y : z] \mapsto [(x, y, z)A]\,,$$

where $[v]$ on the right-hand side means the point in $\mathbb{RP}^2$ corresponding to the line spanned by $v \in \mathbb{R}^3$. (This formula indeed defines a map, because $[(\lambda x, \lambda y, \lambda z)A] = [\lambda(x, y, z)A] = [(x, y, z)A]$, so the right-hand side does not depend on the choice of homogeneous coordinates x, y, z of a point in $\mathbb{RP}^2$.) Such mappings $\mathbb{RP}^2 \to \mathbb{RP}^2$ are known as projective transformations. Invertible projective transformations (which correspond to invertible matrices A) are diffeomorphisms, because in coordinates they are given by rational functions, and rational functions are smooth wherever they are defined. Furthermore, the inverse of a projective transformation is also projective and hence smooth. So, there is a diffeomorphism of $\mathbb{RP}^2$ to itself taking our set to $x^2 + y^2 = z^2$. The latter is completely contained in the chart $z \neq 0$ and is given in the corresponding coordinates by $x_1^2 + x_2^2 = 1$, which is a submanifold. But diffeomorphisms take submanifolds to submanifolds, so our initial set is also a submanifold.

This argument also shows that the submanifold in question is diffeomorphic to S^1.