MATH534A, Solutions for Exam 2

Problem 1. Show that the map $S^2 \to \mathbb{R}^4$ given by

$$(x, y, z) \mapsto (xy, xz, y^2 - z^2, 2yz) \tag{1}$$

is an immersion.

Solution. Denote this map $S^2 \to \mathbb{R}^4$ by F. By the way it is defined, it is a restriction to S^2 of the map $\hat{F} \colon \mathbb{R}^3 \to \mathbb{R}^4$ given by same formula (1). So we have

$$d_pF = (d_p\hat{F})|_{T_pS^2} \quad \forall\, p \in S^2.$$

We need to show that the map

$$d_pF \colon T_pS^2 \to \mathbb{R}^4$$

is injective for any $p \in S^2$. To that end, we first study the map $d_p\hat{F} \colon \mathbb{R}^3 \to \mathbb{R}^4$. Clearly, if the latter is injective, then its restriction to T_pS^2, that is the map d_pF, is injective as well. The matrix of $d_p\hat{F}$, written in natural bases of $\mathbb{R}^3$ and $\mathbb{R}^4$, is

$$\begin{pmatrix} y & x & 0 \\ z & 0 & x \\ 0 & 2y & -2z \\ 0 & 2z & 2y \end{pmatrix}. \tag{2}$$

The 3×3 minor spanned by the rows 1,3,4 is equal to $2y(y^2 + z^2)$. This is not zero provided that $y \neq 0$. Similarly, the minor spanned by the last three rows is equal to $2z(y^2 + z^2)$, which is not zero for $z \neq 0$. So, if at least one of the coordinates y, z does not vanish, then $d_p\hat{F}$ is injective, and so is d_pF. Therefore, it remains to consider the case when $y = z = 0$. For points on the sphere, this means $x = \pm 1$. The tangent plane to S^2 at both points $(0, 0, 1)$ and $(0, 0, -1)$ is spanned by the vectors $\frac{\partial}{\partial y}, \frac{\partial}{\partial z} \in \mathbb{R}^3$. Since d_pF is the restriction of $d_p\hat{F}$ to T_pS^2, it follows that the matrix of d_pF in the basis $\frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ is obtained from matrix (2) of $d_p\hat{F}$ by taking the two last columns (to compute the matrix of d_pF we only need to choose a basis in its domain T_pS^2, because its codomain $\mathbb{R}^4$ has a natural basis). So, the matrix of $d_p\hat{F}$ is

$$\begin{pmatrix} x & 0 \\ 0 & x \\ 2y & -2z \\ 2z & 2y \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This matrix has rank 2, so d_pF is injective at $(0, 0, \pm 1)$, as desired.

Problem 2. Consider the map $\phi \colon \mathbb{R}^2 \to S^2$ given by

$$x = \cos(u)\cos(v), \quad y = \cos(u)\sin(v), \quad z = \sin(u). \tag{3}$$

Find the 1-form $\phi^*\alpha$, where α is the restriction to S^2 of the 1-form $xdy - ydx$.

Solution 1. Observe that (3) is not a coordinate representation of ϕ, because x, y, z are not coordinates on S^2. Instead, these formulas provide a coordinate representation of the composition $i \circ \phi$, where $i \colon S^2 \to \mathbb{R}^3$ is the inclusion mapping. Furthermore, we have

$$\phi^*\alpha = \phi^* i^*(xdy - ydx) = (i \circ \phi)^*(xdy - ydx),$$

where in the first equality we used that restriction is the same as pull-back by the inclusion map, while in the second equality we used the relation

$$(a \circ b)^* = b^* a^*,$$

which is true for any smooth maps a, b (such that the domain of a coincides with the codomain of b). The latter relation is true because

$$d(a \circ b) = da \circ db,$$

while the pull-back is defined as the dual of the differential.

So, since the coordinate representation of $i \circ \phi$ is (3), it follows that the desired form can be obtained from $xdy - ydx$ by plugging in $x = \cos(u)\cos(v), y = \cos(u)\sin(v)$. A direct computation gives

$$(i \circ \phi)^*(xdy - ydx) = \cos^2(u)dv.$$

Solution 2. We compute the desired form step by step, first by restricting to S^2, then by pulling back. In the open northern hemisphere, we have (graph) coordinates s, t on S^2 such that the inclusion mapping $S^2 \to \mathbb{R}^3$ is given by $x = s, y = t, z = \sqrt{1 - s^2 - t^2}$. The restriction α of $xdy - ydx$ to the northern hemisphere is obtained by a direct substitution $x = s, y = t$, which gives $\alpha = sdt - tds$. To compute $\phi^*\alpha$, we plug in the coordinate representation of the map ϕ, which is $s = \cos(u)\cos(v), t = \cos(u)\sin(v)$. This gives $\phi^*\alpha = \cos^2(u)dv$ for all points $p \in \mathbb{R}^2$ such that $\phi(p)$ is in the open northern heimisphere. Analogously, $\phi^*\alpha = \cos^2(u)dv$ for all points $p \in \mathbb{R}^2$ such that $\phi(p)$ is in the open southern heimisphere (the only difference between the northern and southern hemispheres is the sign of z, but z does not enter the expression $xdy - ydx$). So, $\phi^*\alpha = \cos^2(u)dv$, possibly except for the points mapping to the equator, i.e. points with $\sin(u) = 0$. At the same time, the form $\phi^*\alpha$ is smooth and in particular continuous (i.e. its coefficients in the basis du, dv are continuous functions), so we must have $\phi^*\alpha = \cos^2(u)dv$ everywhere.

Solution 3. By definition, for any $\xi \in T_p\mathbb{R}^2$, we have

$$\phi^*\alpha(\xi) = \alpha(d_p\phi(\xi)).$$

To find the coordinates of $\phi^*\alpha$ in the basis du, dv, we need to compute the values of that form on the vectors $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$. So, the du coefficient is

$$\phi^*\alpha(\frac{\partial}{\partial u}) = \alpha(d_p\phi(\frac{\partial}{\partial u})) = \alpha(-\sin(u)\cos(v)\frac{\partial}{\partial x} - \sin(u)\sin(v)\frac{\partial}{\partial y} + \cos(u)\frac{\partial}{\partial z}).$$

Here we used that $d_p\phi(\frac{\partial}{\partial u})$ is given by the first column of the Jacobian matrix of ϕ.

Further, by definition of restriction, the value of α at a tangent vector to S^2 is the same as the value of $xdy - ydx$ at the same vector, so

$$\begin{aligned}\alpha(-\sin(u)\cos(v)\frac{\partial}{\partial x} &- \sin(u)\sin(v)\frac{\partial}{\partial y} + \cos(u)\frac{\partial}{\partial z}) \\ &= (xdy - ydx)(-\sin(u)\cos(v)\frac{\partial}{\partial x} - \sin(u)\sin(v)\frac{\partial}{\partial y} + \cos(u)\frac{\partial}{\partial z}) \\ &= y\sin(u)\cos(v) - x\sin(u)\sin(v).\end{aligned}$$

Since we are applying α to a tangent vector at the point $\phi(p)$, we should evaluate the coefficients of α at that point. So, in the latter expression we set $x = \cos(u)\cos(v), y = \cos(u)\sin(v)$, which shows that the du coefficient of $\phi^*\alpha$ vanishes. A similar computation shows that the dv coefficient is $\cos^2(u)$, so $\phi^*\alpha = \cos^2(u)dv$.

Problem 3. Let M be a compact manifold, and let $C \subset M$ be its closed subset. Let also v be a smooth vector field defined on some open set $U \supset C$. Show that there exists a smooth vector field $\hat{v}$ on M such that $\hat{v}|_C = v$.

Solution. Let $U_1 = U$, $U_2 = M \setminus C$. Then U_1, U_2 is an open cover of M. So, there exists a partition of unity f_1, f_2 subordinate to this cover. Define

$$\hat{v}(p) = \begin{cases} f_1(p)v(p), & \text{if } p \in U_1, \\ 0, & \text{if } p \notin U_1. \end{cases}$$

Then $\hat{v}$ is a vector field, and at any point $p \in C$ we have

$$\hat{v}(p) = f_1(p)v(p) = v(p),$$

where we used that $f_1 + f_2 = 1$, and $f_2 = 0$ outside of $U_2 = M \setminus C$, i.e. in C. So, it remains ti show that $\hat{v}$ is smooth. This is so for points $p \in U_1$ since in U_1 we have $\hat{v} = f_1 v$, while both f_1 and v are smooth. On the other hand, if $p \notin U_1$, then $p \notin \operatorname{supp} f_1$ (since $\operatorname{supp} f_1 \subset U_1$ by definition of partition of unity subordinate to a cover), so $p \in M \setminus \operatorname{supp} f_1$. The latter set is open (since $\operatorname{supp} f_1$ is, by definition, closed) and for any its point we have $\hat{v} = 0$ (since $f_1 = 0$ outside $\operatorname{supp} f_1$ by definition of support). So, there is an open neighborhood of p where $\hat{v} = 0$, which means that $\hat{v}$ is smooth at p, as desired.