MATH534B, Recommended Problems (Last Updated on April 25, 2019)

1. Show that the relation of homotopy among continuous maps $X \to Y$, where X and Y are fixed topological spaces, is an equivalence relation.

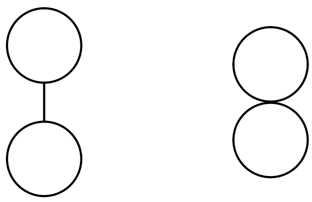
2. Show that homotopy equivalence of topological spaces is an equivalence relation.

3. Assume that X is a topopological space, and $\gamma_1, \gamma_2 \colon [0,1] \to X$ are two paths in X such that $\gamma_1(1) = \gamma_2(0)$. Assume also that $\tilde{\gamma}_1$ is homotopic to γ_1 with fixed endpoints, while $\tilde{\gamma}_2$ is homotopic to γ_2 with fixed endpoints. Show that the concatenations $\tilde{\gamma}_1\tilde{\gamma}_2$, $\gamma_1\gamma_2$ are homotopic with fixed endpoints.

4. Assume that X and Y are homotopy equivalent topological spaces. Show that if X is path connected, then so is Y.

5. Assume that a topological space X is contractible. Show that every point $x \in X$ is a deformation retract of X.

6. Parametrizing line intervals and circles in the following figure, give an accurate proof that the depicted topological spaces are homotopy equivalent:



7. Let $\gamma_1, \gamma_2, \gamma_3 \colon [0,1] \to X$ be paths in a topological space X. Assume also that $\gamma_1(1) = \gamma_2(0)$, $\gamma_2(1) = \gamma_3(0)$. Shows that the paths $(\gamma_1 \cdot \gamma_2) \cdot \gamma_3$ and $\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)$ are homotopic with fixed endpoints. Here $\gamma_i \cdot \gamma_j$ stands for the concatenation of paths,

$$(\gamma_i \cdot \gamma_j)(t) = \begin{cases} \gamma_i(2t) & \text{if } t \le \dfrac{1}{2}, \\ \gamma_j(2t-1) & \text{if } t \ge \dfrac{1}{2}. \end{cases}$$

8. Assume that X and Y are topological spaces, and let $K \subset X$. Two continuous maps $f, g \colon X \to Y$ are said to be *homotopic relative to* K if

   (a) $f(k) = g(k)$ for any $k \in K$.

   (b) There exists a homotopy $H \colon X \times [0,1] \to Y$ between f and g such that $F(k,t) = g(k) = h(k)$ for any $k \in K, t \in [0,1]$.

   For example, homotopy with fixed endpoints between paths $[0,1] \to X$ is the same as homotopy relative to $\{0,1\}$.

   Let $p \in S^1$ be any point. Show that the fundamental group $\pi_1(X, x_0)$ can be defined as the set of maps $f \colon S^1 \to X$ with $f(p) = x_0$ modulo homotopies relative to $\{p\}$. Describe the group operation on $\pi_1(X, x_0)$ in terms of this definition.

9. Let $\gamma \colon [0,1] \to S^1$ be a path. Say that $\alpha \colon [0,1] \to \mathbb{R}$ is a continuous determination of the polar angle of $\gamma(t)$ if $\gamma(t) = (\cos\alpha(t), \sin\alpha(t))$ for any $t \in [0,1]$. Show that if α_1, α_2 are two continuous determinations of the polar angle of $\gamma(t)$, then $\alpha_1(t) - \alpha_2(t) = 2\pi k$, where k is an integer independent of t.

10. Let γ be a loop in S^1 based at x_0, i.e. a continuous map $\gamma\colon [0,1] \to S^1$ such that $\gamma(0) = \gamma(1) = x_0$. Let also $\alpha\colon [0,1] \to \mathbb{R}$ be a continuous determination of the polar angle of $\gamma(t)$, which means that $\gamma(t) = (\cos\alpha(t), \sin\alpha(t))$ for any $t \in [0,1]$. Recall that the number of times $\gamma(t)$ goes around S^1 is defined by the formula
$$N(\gamma) = \frac{\alpha(1) - \alpha(0)}{2\pi}.$$
Prove that if γ is smooth in $(0,1)$, then $N(\gamma)$ is given by
$$N(\gamma) = \frac{1}{2\pi}\int_0^1 \gamma^* d\phi,$$
where the 1-form $d\phi$ on S^1 is given by $d\phi = (xdy - ydx)|_{S^1}$.

11. Let $\gamma\colon [0,1] \to S^1$ be a path (not necessarily a loop) in S^1, and let $\alpha\colon [0,1] \to \mathbb{R}$ be a continuous determination of the polar angle of $\gamma(t)$, which means that $\gamma(t) = (\cos\alpha(t), \sin\alpha(t))$ for any $t \in [0,1]$. Define
$$\Delta(\gamma) = \alpha(1) - \alpha(0).$$
Show that
    (a) If $\gamma_1, \gamma_2\colon [0,1] \to S^1$ are homotopic with fixed endpoints, then $\Delta(\gamma_1) = \Delta(\gamma_2)$.
    (b) If $\gamma_1, \gamma_2\colon [0,1] \to S^1$ are such that $\gamma_2(0) = \gamma_1(1)$, then $\Delta(\gamma_1 \cdot \gamma_2) = \Delta(\gamma_1) + \Delta(\gamma_2)$.

12. Let $\gamma_1, \gamma_2\colon [0,1] \to X$ be paths in a topological space X such that $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(1) = \gamma_2(1)$. Assume that the loop $\gamma_1 \cdot \gamma_2^{-1}$ (where $\gamma_2^{-1}(t) = \gamma_2(1-t)$) is trivial in $\pi_1(X, \gamma_1(0))$. Show that γ_1, γ_2 are homotopic with fixed endpoints.

13. Consider the set of paths in S^1 starting at a given point x_0. Say that two such paths are equivalent if they have the same endpoint and are homotopic with fixed endpoints. Show that the mapping Δ defined in Exercise 11 is a bijection between such equivalence classes and $\mathbb{R}$.

14. Let γ be a loop in $\mathbb{R}^2 \setminus (0,0)$ based at a certain point x_0. Let also $\alpha\colon [0,1] \to \mathbb{R}$ be a continuous determination of the polar angle of $\gamma(t)$, which means that $\gamma(t) = (r(t)\cos\alpha(t), r(t)\sin\alpha(t))$ for any $t \in [0,1]$ and an appropriate function $r(t)$. Define the number of times $\gamma(t)$ goes around the origin by the formula
$$N(\gamma) = \frac{\alpha(1) - \alpha(0)}{2\pi}.$$
Show that the mapping N is well-defined on $\pi_1(\mathbb{R}^2 \setminus (0,0), x_0)$ and maps the latter group isomorphically to $\mathbb{Z}$.

15. Let γ be a smooth loop in $\mathbb{C} \setminus 0$. Show that number of times $\gamma(t)$ goes around the origin (defined in the preceding exercise) is given by the formula
$$N(\gamma) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z}.$$

16. Identify the circle with the set of complex numbers with absolute value 1. For the map $f\colon S^1 \to S^1$ given by $f(z) = z^n$, where $n \in \mathbb{Z}$, compute the induced homomorphism on $\pi_1(S^1, 1)$.

17. Prove Brouwer's fixed point theorem in the 1-dimensional case: every continuous function $f\colon [0,1] \to [0,1]$ has a fixed point, i.e. a point $x \in [0,1]$ such that $f(x) = x$.

18. Does every continuous mapping of an open disc to itself have a fixed point?

19. For a function $f\colon \mathbb{C} \to \mathbb{C}$, let

$$\gamma_{f,r}(t) = \frac{f(re^{2\pi it})}{f(r)},$$

and let $N(f, r)$ be the class of $\gamma_{f,r}$ in $\pi_1(\mathbb{C}^*, 1)$ (it is well defined provided that f does not vanish at the circle of radius r centered at the origin). Note that $\pi_1(\mathbb{C}^*)$ is isomorphic to $\mathbb{Z}$ and we can fix this isomorphism by requiring that a loop going around the origin once in the counterclockwise direction corresponds to 1. So, $N(f, r)$ can be regarded as an integer. Compute

$$\lim_{r\to\infty} N(f, r),$$

where f is a rational function with numerator of degree p and denominator of degree q.

20. Show that for a complex-differentiable function f, the quantity $N(f, r)$ defined in the previous exercise is given by

$$N(f, r) = \frac{1}{2\pi i}\int_{|z|=r} \frac{f'(z)}{f(z)} dz.$$

Hence show that if f is meromorphic in the disc D_r of radius r centered at the origin and has no zeros or poles on the boundary of that disc, then $N(f, r)$ is equal to the number of zeros of f in D_r minus the number of poles of f in D_r (each zero and pole being counted with multiplicity).

21. Prove the following lemma that we used to compute the fundamental groups of spheres. Let $\gamma\colon [0, 1] \to X$ be a continuous mapping, and let $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$ be points in $(0, 1)$. Let also $\gamma_i\colon [a_i, b_i] \to X$ be continuous mappings such that $\gamma_i(a_i) = \gamma(a_i)$ and $\gamma_i(b_i) = \gamma(b_i)$. Define a new mapping $\tilde{\gamma}\colon [0, 1] \to X$ by

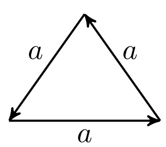
$$\tilde{\gamma}(t) = \begin{cases} \gamma_i(t) & \text{if there is } i \text{ such that } a_i < t < b_i, \\ \gamma(t) & \text{otherwise.} \end{cases}$$

Prove that $\tilde{\gamma}$ is continuous. Also show that it does not have to be true if the number of intervals $[a_i, b_i]$ is infinite.

22. Exercise 1 for Section 1.1 in Hatcher.

23. Show that the torus with a hole (the hole is open and homeomorphic to a disc) does not retract to its boundary circle.

24. Compute the fundamental group of a solid triangle whose sides are identified as shown in the figure (here we use the standard convention: all sides with the same label are identified according to the direction of the arrows):



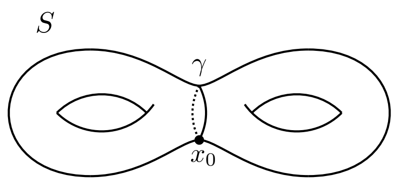
25. Prove that the space from the previous exercise is not a topological manifold.

26. Prove that quotient of $\mathbb{Z} * \mathbb{Z}$ by its commutator subgroup is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

27. Let $\langle x\rangle$, $\langle y\rangle$, $\langle z\rangle$ be free Abelian groups generated by elements x, y, z respectively. Show that the elements $xyz, xzy \in \langle x\rangle * \langle y\rangle * \langle z\rangle$ are not conjugate to each other. Hint: show that if G is any group, and $g_1, g_2, g_3 \in G$, then there is a homomorphism $\phi\colon \langle x\rangle * \langle y\rangle * \langle z\rangle \to G$ such that $\phi(x) = g_1, \phi(y) = g_2, \phi(z) = g_3$. So, if xyz, xzy are conjugate to each other in $\langle x\rangle * \langle y\rangle * \langle z\rangle$, then $g_1g_2g_3$ and $g_1g_3g_2$ are conjugate to each other in G. Therefore, it suffices to find any group G and any three elements $g_1, g_2, g_3 \in G$ such that $g_1g_2g_3$ and $g_1g_3g_2$ are not conjugate to each other.

28. Let K be the unit square, and let $f\colon K \to X$ be its map to a topological space X. Assume also that where $A, B \subset X$ are open subset such that $A \cup B = X$. Show that there exists a finite rectangular grid on K such that each (closed) rectangle K_{ij} of the grid satisfies $f(K_{ij}) \subset A$ or $f(K_{ij}) \subset B$. By a rectangular grid we mean the following: choose two partitions $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$, $0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = 1$ of the unit interval. Then the associated grid on K is the partition
$$K = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} K_{ij}$$
with $K_{ij} = [t_{i-1}, t_i] \times [s_{j-1}, s_j]$.

29. The surface S shown in the figure is obtained from two tori by removing a disc from each of them and gluing the resulting surfaces along the boundary (this operation is known as taking the *connected sum* of two tori). This surface is often referred to as *the pretzel*.



Compute $\pi_1(S, x_0)$. Determine whether the class of γ in $\pi_1(S, x_0)$ is equal to the identity.

30. Compute the fundamental group of the union of edges of a cube.

31. Compute the fundamental group of the 3-space with 3 straight lines removed. How does the answer depend on the mutual position of the lines?

32. Let $p\colon \tilde{X} \to X$ be a local homeomorphism. Assume also that $\tilde{X}$ is compact. Show that p is a finite covering.

33. Identify the 2-dimensional torus T^2 with $S^1 \times S^1$. Then each point in T^2 can be written as (p, q) with $p, q \in S^1$. Let ϕ be the polar angle of p, and ψ be the polar angle of q. Then ϕ and ψ can be thought of as coordinates on the torus (longitude and latitude), both defined modulo 2π. Consider the following map of T^2 to itself:
$$(\phi, \psi) \mapsto (\phi + \psi, \phi + 3\psi).$$
Show that this map is a covering, compute its degree, and describe all deck transformations.

34. Prove that the mapping $\mathbb{C} \setminus \{\pm 1, \pm 2\} \to \mathbb{C} \setminus \{\pm 2\}$ given by $z \mapsto z^3 - 3z$ is a covering. Show that it admits no non-trivial deck transformations.

35. Construct a triangulation and compute the simplicial homology for the following spaces: closed interval, closed two-dimensional disk, cylinder, Moebius band.

36. Consider the union of faces of a tetrahedron as a triangulation of the sphere S^2. Compute the simplicial homology of the sphere using that triangulation.

37. The *Euler characteristic* of a topological space X is defined as
$$\chi(X) = \beta_0 - \beta_1 + \beta_2 - \ldots,$$
where β_i's are Betti numbers and X is such that the sum in the right-hand side has only finite many non-zero terms. Prove that for a topological space X that admits a triangulation with finite number of simplices one has
$$\chi(X) = \text{number of 0-simplices} - \text{number of 1-simplices} + \text{number of 2-simplices} - \ldots.$$

38. Prove that the Euler characteristic of a 2-sphere is well-defined (i.e. does not depend on the choice of the triangulation) and is equal to 2. Prove Euler's formula for a convex polytope:

$$\text{number of vertices} - \text{number of edges} + \text{number of faces} = 2.$$

39. Prove that singular homology groups of homeomorphic spaces are isomorphic.

40. Prove the *five lemma*: in the commutative diagram of Abelian groups below, if the two rows are exact, while the two leftmost and two rightmost vertical arrows are isomorphisms, then the middle vertical arrow is an isomorphism as well.

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

41. Prove that a short exact sequence $0 \to A_n \to B_n \to C_n \to 0$ of chain complexes gives rise to the following long exact sequence of homology groups:

$$\cdots \to H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to \ldots$$

42. Let X be a topological space, and let $Y \subset X$ be an embedded circle which is a boundary in the following sense: the map $i_*\colon H_1(Y) \to H_1(X)$, induced by the inclusion $Y \to X$, takes the fundamental class of Y to 0. Assume also that Y has an open neighborhood in X which deformation retracts to Y. Express the homology groups of X/Y in terms of homology groups of X.

43. Consider the region R of $\mathbb{R}^3$ bounded by the pretzel from Problem 29 (loosely speaking, R consists of points that are inside the pretzel). Let X be the space obtained from two copies of R by identifying their boundaries (via the identity map).

    (a) Prove that X is a topological 3-dimensional manifold.

    (b) Using the Mayer-Vietoris sequence, compute the homology groups of X (with coefficients in $\mathbb{Z}$).

44. Let $D \subset \mathbb{R}^n$ be an open subset bounded by a smooth hypersurface S. Let also V_n be the volume of a unit ball in $\mathbb{R}^n$. Prove that the expression

$$\frac{1}{nV_n}\int_S \frac{\sum_{i=1}^n (-1)^{i+1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n}{(\sum_{i=1}^n x_i^2)^{n/2}}$$

is equal to 1 if the region D contains the origin, and 0 otherwise (we assume that S does not contain the origin, so the above integral is well-defined).

45. Prove that for a 2π-periodic function $f(x)$ one has

$$\int_a^{a+2\pi} f(x)dx = \int_0^{2\pi} f(x)dx.$$

46. Prove that de Rham cohomology groups of diffeomorphic manifolds are isomorphic.

47. Let $\phi\colon M \to N$ be a diffeomorphism between compact oriented n-dimensional manifolds. Show that for any top degree-form $\omega \in \Omega^n(N)$ one has

$$\int_N \omega = \pm \int_M \phi^*\omega.$$

Moreover, the sign in the right-hand side is positive if ϕ is orientation-preserving, and negative otherwise.

48. Let M be a manifold, and let $U, V \subset M$ be open subsets such that the closure $\bar{V}$ is contained in U. Let also $f\colon U \to \mathbb{R}$ be a smooth function. Show that there exists a smooth function $F\colon M \to \mathbb{R}$ such that $F|_V = f$.