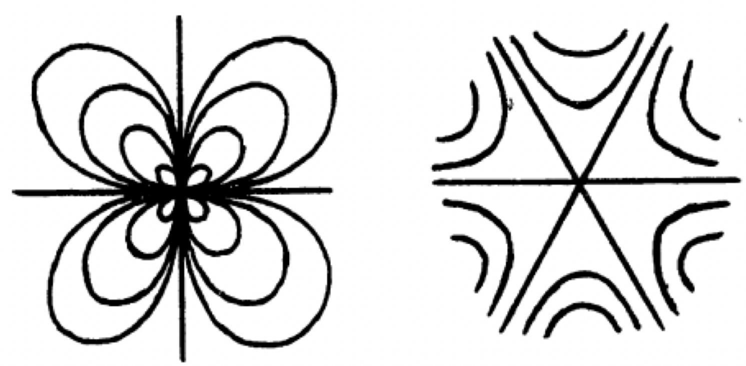
## Problem 32: Index of a singular point of a gradient vector field

**Theorem**: *In dimension two, $ind \leq 1$.*

**Proof**:

$$ind = \frac{e-h}{2} + 1,$$

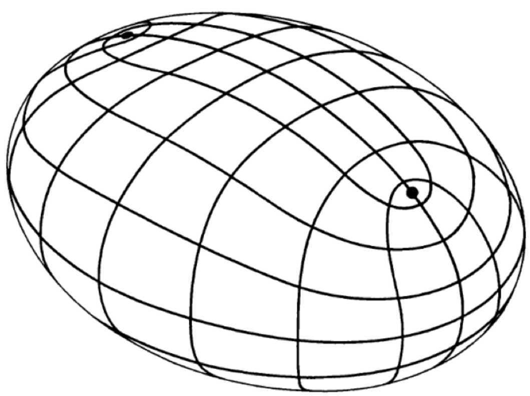
where $e$ and $h$ is the number of elliptic and hyperbolic sectors. For gradients, $e = 0$, QED



**Loewner's cojecture.** Consider the vector field $(\partial_x + i\partial_y)^n F(x, y)$, where $F$ is a real smooth function. For example, if $n = 1$, this is $\nabla F$, and if $n = 2$, the field is $(F_{xx} - F_{yy}, 2F_{xy})$.

**Conjecture**: *The index of an isolated singular point of such field does not exceed $n$.*

The $n = 2$ case implies the Caratéodory conjecture: *a smooth surface in $\mathbf{R}^3$ diffeomorphic to the sphere has at least two distinct umbilic points.*

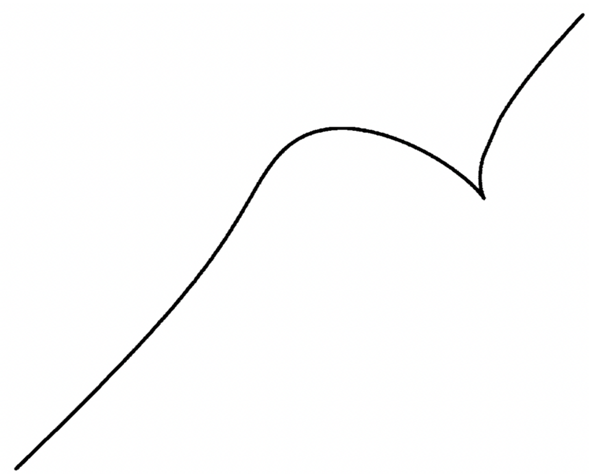


The umbilic points of a convex surface are the conformal points of the Hamiltonian vector field of its support function $S^2 \to \mathbf{R}$, prompting

**Conjecture**: *An area preserving diffeomorphism of a round* $S^2$ *possesses at least two distinct conformal points.*

## Problem 34: Curve $y = x^3$ in $\mathbf{RP}^2$

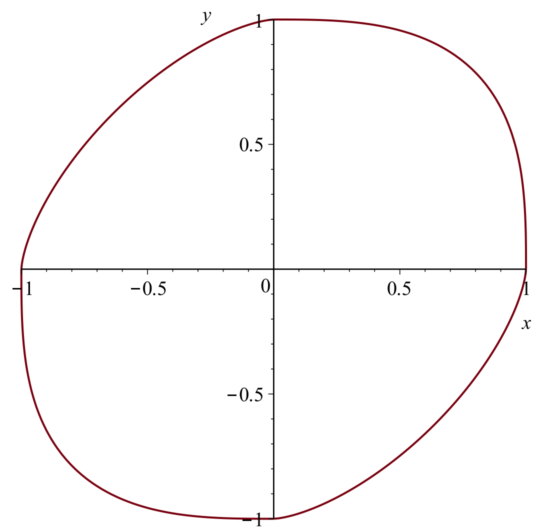
*This curve is projectively self-dual*: it has a cusp at infinity, dual to its inflection point at the origin.



**Definition**: Let $\gamma \subset \mathbf{RP}^2$ be a curve, perhaps with cusps, and $\gamma^* \subset (\mathbf{RP}^2)^*$ be the dual curve. Then $\gamma$ is projectively self-dual if there is a projective map $\varphi : \mathbf{RP}^2 \to (\mathbf{RP}^2)^*$ that takes $\gamma$ to $\gamma^*$.

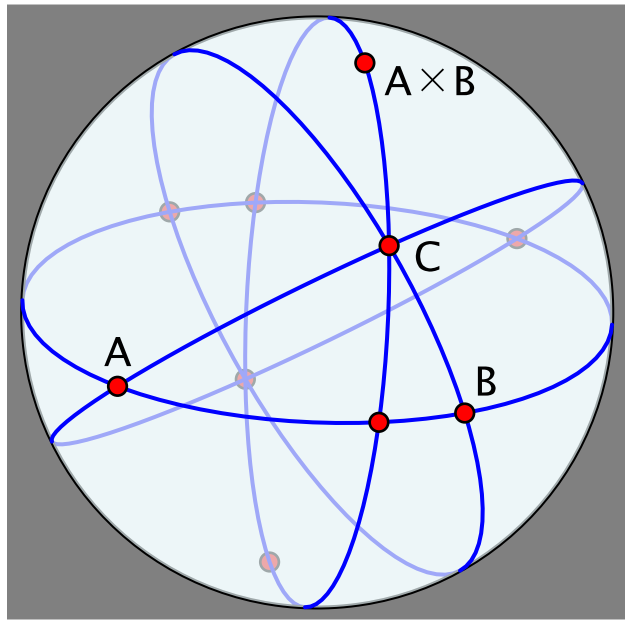
An obvious example: conics. A complete description of self-dual curves is not known (unlike self-dual polygons); it is Problem No 1994-17 in Arnold's book of problems.

**Example**: Radon curve. For example, $p = 3, q = 3/2$




$$x^p + y^p = 1 \text{ for } xy > 0, \; x^q + y^q = 1 \text{ for } xy < 0, \text{ with } \frac{1}{p} + \frac{1}{q} = 1.$$

## Problem 42: Are the altitudes of a triangle concurrent?




**Spherical proof**:

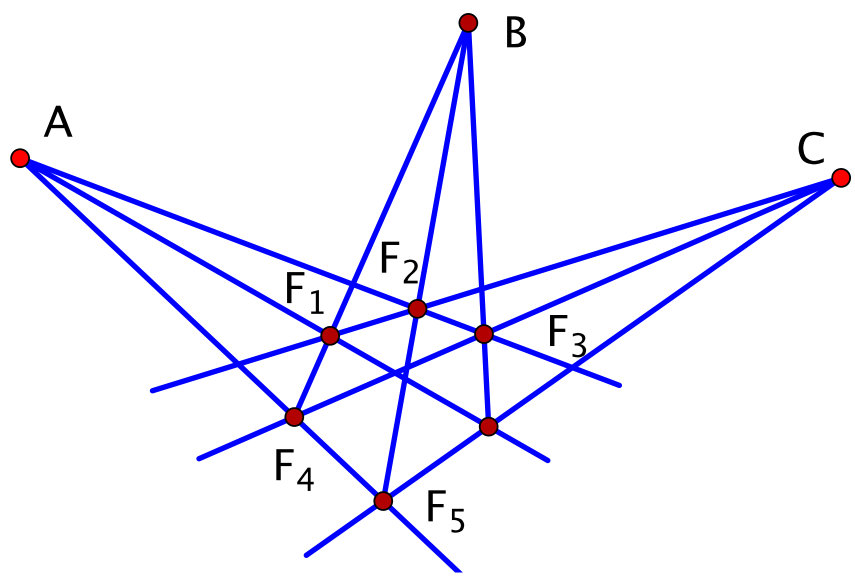
$$(A \times B) \times C + (B \times C) \times A + (C \times A) \times B = 0,$$

the Jacobi identity in $so(3)$!

Likewise in $H^2$, with the Lie algebra $sl(2, \mathbf{R})$ (but not in $\mathbf{R}^2$).

T. Tomihisa: the dual Pappus theorem as an identity in $sl(2)$

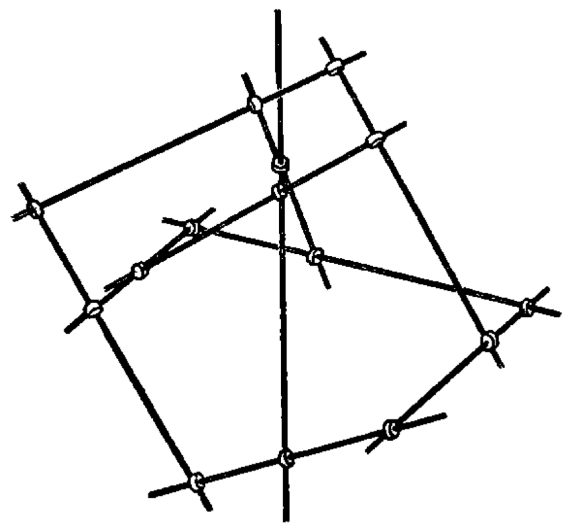
$$[F_1, [[F_2, F_3], [F_4, F_5]]] + [F_3, [[F_2, F_5], [F_4, F_1]]] + [F_5, [[F_2, F_1], [F_4, F_3]]] = 0.$$



Skewer (common normal) variant [Petersen-Hjelmslev-Morley, 1897]:
*Given three lines* $a, b, c$ *in* $\mathbf{R}^3$ *(or* $S^3$*, or* $H^3$*), the lines*

$$S(S(a,b),c),\ \ S(S(b,c),a),\quad \text{and}\quad S(S(c,a),b)$$

*share a skewer.*



Equivalently, the common normals of the opposite sides of a rectangular hexagon have a common normal.

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