Problem 32: Index of a singular point of a gradient vector field

**Theorem:** *In dimension two, $ind \leq 1$.*

**Proof:**

$$ind = \frac{e - h}{2} + 1,$$

where $e$ and $h$ is the number of elliptic and hyperbolic sectors. For gradients, $e = 0$, QED
Loewner’s conjecture. Consider the vector field $(\partial_x+i\partial_y)^nF(x,y)$, where $F$ is a real smooth function. For example, if $n = 1$, this is $\nabla F$, and if $n = 2$, the field is $(F_{xx} - F_{yy}, 2F_{xy})$.

Conjecture: The index of an isolated singular point of such field does not exceed $n$.

The $n = 2$ case implies the Caratéodory conjecture: a smooth surface in $\mathbb{R}^3$ diffeomorphic to the sphere has at least two distinct umbilic points.
The umbilic points of a convex surface are the conformal points of the Hamiltonian vector field of its support function $S^2 \to \mathbb{R}$, prompting

**Conjecture:** An area preserving diffeomorphism of a round $S^2$ possesses at least two distinct conformal points.
Problem 34: Curve $y = x^3$ in $\mathbb{RP}^2$

This curve is projectively self-dual: it has a cusp at infinity, dual to its inflection point at the origin.

Definition: Let $\gamma \subset \mathbb{RP}^2$ be a curve, perhaps with cusps, and $\gamma^* \subset (\mathbb{RP}^2)^*$ be the dual curve. Then $\gamma$ is projectively self-dual if there is a projective map $\varphi : \mathbb{RP}^2 \rightarrow (\mathbb{RP}^2)^*$ that takes $\gamma$ to $\gamma^*$. 
An obvious example: conics. A complete description of self-dual curves is not known (unlike self-dual polygons); it is Problem No 1994-17 in Arnold’s book of problems.

**Example:** Radon curve. For example, $p = 3, q = 3/2$

\[
x^p + y^p = 1 \text{ for } xy > 0, \quad x^q + y^q = 1 \text{ for } xy < 0, \text{ with } \frac{1}{p} + \frac{1}{q} = 1.
\]
Problem 42: Are the altitudes of a triangle concurrent?

Spherical proof:

\[(A \times B) \times C + (B \times C) \times A + (C \times A) \times B = 0,\]

the Jacobi identity in $so(3)$!
Likewise in $H^2$, with the Lie algebra $sl(2, \mathbb{R})$ (but not in $\mathbb{R}^2$).

T. Tomihisa: the dual Pappus theorem as an identity in $sl(2)$

$$[F_1, [[F_2, F_3], [F_4, F_5]]] + [F_3, [[F_2, F_5], [F_4, F_1]]] + [F_5, [[F_2, F_1], [F_4, F_3]]] = 0.$$
Skewer (common normal) variant [Petersen-Hjelmslev-Morley, 1897]: Given three lines $a, b, c$ in $\mathbb{R}^3$ (or $S^3$, or $H^3$), the lines

$$S(S(a, b), c), S(S(b, c), a), \text{ and } S(S(c, a), b)$$

share a skewer.

Equivalently, the common normals of the opposite sides of a rectangular hexagon have a common normal.
References


