ADDENDUM: ÉTALE DÉVISSAGE, DESCENT AND PUSHOUTS OF STACKS

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Abstract. Using Nisnevich coverings and a Hilbert stack of stacky points, we prove étale dévissage results for non-representable étale and quasi-finite flat coverings. We give applications to absolute noetherian approximation of algebraic stacks and compact generation of derived categories.

1. Introduction

In [Ryd11a Thm. D & 6.1], dévissage results were proved for representable quasi-finite flat and étale morphisms. We will show how these results may be extended to the non-representable situation using Nisnevich coverings and a Hilbert stack of stacky points.

We apply these results to weaken the separation hypotheses from the approximation results for algebraic stacks that appeared in [Ryd15] and the compact generation result for derived categories of quasi-coherent sheaves on Deligne–Mumford stacks that appeared in [HR17 Thm. A].

The results of this article have already been used in [HK17]. We also expect further applications arising from the work of [AHR15, AHR14] on the local structure of stacks near points with linearly reductive stabilizers, where non-representable étale coverings naturally arise (see Remark 7.6).

Before stating our main result, we require some notation. Fix an algebraic stack $S$. If $P_1, \ldots, P_r$ is a list of properties of morphisms of algebraic stacks over $S$, let $\text{Stack}_{P_1, \ldots, P_r}/S$ denote the full 2-subcategory of the 2-category of algebraic stacks over $S$ whose objects are those $(x : X \to S)$ such that $x$ has properties $P_1, \ldots, P_r$. The following abbreviations will be used: ét (étale), qff (quasi-finite flat), sep (separated), fp (finitely presented), rep (representable), and sep ∆ (separated diagonal). Throughout, we let $E \subseteq \text{Stack}/S$ be one of the following 2-subcategories:

$$
\text{Stack}_{\text{repr}, \text{sep}, \text{fp}, \text{ét}}/S \subseteq \text{Stack}_{\text{sep}, \text{fp}, \text{ét}}/S \subseteq \text{Stack}_{\text{sep}, \text{fp}, \text{ét}, \text{rep}}/S
$$

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Our improvement of [Ryd11a Thm. D & 6.1] is the following theorem.

**Theorem D’** (Étale or quasi-finite flat dévissage). Let $S$ be a quasi-compact and quasi-separated algebraic stack and let $E$ be as above. Let $(T’ \xrightarrow{f} T) \in E$ be surjective (resp. surjective and representable) and let $D \subseteq E$ be a full 2-subcategory satisfying the following three conditions:

- **(D1)** if $(X’ \to X) \in E$ is étale and $X \in D$, then $X’ \in D$;
- "
(D2) if $(X' \to X) \in \mathbf{E}$ is proper (resp. finite) and surjective and $X' \in \mathbf{D}$, then $X \in \mathbf{D}$; and

(D3) if $(U \to X)$, $(X' \to X) \in \mathbf{E}$, where $i$ is an open immersion and $f$ is étale and an isomorphism over $X \setminus U$, then $X \in \mathbf{D}$ whenever $U$, $X' \in \mathbf{D}$.

If $T' \in \mathbf{D}$, then $T \in \mathbf{D}$.


Note that if $(X' \to X) \in \mathbf{E}$ is étale, then there is a canonical factorization $X' \to X'' \to X$ in $\mathbf{E}$ where the first morphism is an étale gerbe and the second morphism is étale. If in addition $(X' \to X)$ is proper, then $X' \to X''$ is a proper étale gerbe and $X'' \to X$ is finite étale.

Note that if $T' \to T$ is representable, then it has separated diagonal. In particular, the advantage of Theorem D′ over [Ryd11a, Thm. D] is the removal of the assumption of representability from $T' \to T$.

The “Induction principle” [Stacks, Tag 08GL] for algebraic spaces is closely related to the dévissage results of Theorem D′. When working with derived categories or K-theory, where locality results are often quite subtle, it is often advantageous to have the strongest possible criteria at your disposal (e.g., [Hal16]). For stacks with quasi-finite diagonal, we also obtain the following Induction principle.

**Theorem E** (Induction principle for stacks with quasi-finite diagonal). Let $S$ be a quasi-compact and quasi-separated algebraic stack. Choose $\mathbf{E} \subseteq \text{Stack}^{qs}_{/S}$ as follows:

1. if $S$ has quasi-finite diagonal, take $\mathbf{E} = \text{Stack}_{\text{sep}, \text{fp}, \text{qff}}^{qs}_{/S}$;
2. if $S$ has quasi-finite and separated diagonal, take $\mathbf{E} = \text{Stack}_{\text{sep}, \text{repr}, \text{fp}, \text{qff}}^{qs}_{/S}$;
3. if $S$ is Deligne–Mumford, take $\mathbf{E} = \text{Stack}_{\text{sep}, \text{et}}^{qs}_{/S}$; and
4. if $S$ is Deligne–Mumford with separated diagonal, take $\mathbf{E} = \text{Stack}_{\text{et}, \text{sep}, \text{fp}, \text{qff}}^{qs}_{/S}$.

Let $\mathbf{D} \subseteq \mathbf{E}$ be a full 2-subcategory satisfying the following properties:

1. if $(X' \to X) \in \mathbf{E}$ is an open immersion and $X \in \mathbf{D}$, then $X' \in \mathbf{D}$;
2. if $(X' \to X) \in \mathbf{E}$ is finite and surjective, then $X \in \mathbf{D}$; and
3. if $(U \to X)$, $(X' \to X) \in \mathbf{E}$, where $i$ is an open immersion and $f$ is étale and an isomorphism over $X \setminus U$, then $X \in \mathbf{D}$ whenever $U$, $X' \in \mathbf{D}$.

Then $\mathbf{D} = \mathbf{E}$. In particular, $S \in \mathbf{D}$.

Proof. Combine Lemma 5.4 with Theorem 1.1. □

We wish to point out that Theorem E relies on the existence of coarse spaces for stacks with finite inertia (i.e., the Keel–Mori Theorem [KM97, Ryd13]). Theorem E in the case of a separated diagonal, was proved in [Hal16, App. B].

**Remark 1.1.** Extending Theorem D′ to covers with non-separated diagonals is possible. The most natural and useful formulation, however, requires 2-stacks and the corresponding notion of 2-Nisnevich coverings. This is analogous to the situation of representable but non-separated coverings, where non-representable Nisnevich coverings naturally appear. See Remark 5.4 for more details.

**Conventions.** We make no a priori separation assumptions on our algebraic stacks, just as in [Stacks].

2. Residual gerbes as intersections

Let $X$ be a quasi-separated algebraic stack (e.g., $X$ noetherian). By [Ryd11a, Thm. B.2], every point of $X$ is algebraic. That is, if $x \in \vert X \vert$, then there is a quasi-affine monomorphism $\mathcal{G}_x \to X$ with image $x$ such that $\mathcal{G}_x$ is an fpqc gerbe,
the residual gerbe. Using the recent approximation result [Ryd16], which depends on the original étale dévissage [Ryd11a], we obtain

Lemma 2.1. Let $X$ be a quasi-separated algebraic stack and let $x \in |X|$ be a point. The residual gerbe $\mathcal{G}_x$ is the limit of an inverse system of immersions $j_\lambda: U_\lambda \hookrightarrow X$ of finite presentation with affine bonding maps.

Proof. There is a locally closed integral substack $Z \hookrightarrow X$ such that $Z$ is a gerbe over an affine scheme $\mathbb{Z}$ and $x$ is the generic point of $Z$ [Ryd11a Thm. B.2]. Let $U \subseteq X$ be a quasi-compact open neighborhood of $Z$ such that $Z \hookrightarrow U$ is a closed immersion. Consider the inverse system $\{W_\lambda \hookrightarrow U\}_{\lambda \in \Lambda}$ of all finitely presented affine immersions $W_\lambda \hookrightarrow U$ such that $x \in |W_\lambda|$. We claim that the inverse limit, i.e., the intersection, is $\mathcal{G}_x$.

Indeed, let $\pi: Z \to \mathbb{Z}$ denote the structure map of the gerbe. Then $\pi(x)$ is the intersection of its affine open neighborhoods $Z_\alpha \subseteq \mathbb{Z}$. Thus $\mathcal{G}_x = \pi^{-1}(\text{Spec } \mathbb{Z}(\pi(x)))$ is the intersection of its relatively affine open neighborhoods $Z_\alpha = \pi^{-1}(Z_\alpha)$, i.e., the open immersions $Z_\alpha \to Z$ are affine. Moreover, for a fixed $\alpha$, we may pick an open quasi-compact substack $U_\alpha \subseteq U$ such that $Z_\alpha = Z \cap U_\alpha$. Since $Z_\alpha \hookrightarrow U_\alpha$ is a closed immersion, we may write $Z_\alpha \hookrightarrow U_\alpha$ as the intersection of closed immersions $Z_{\alpha \beta} \hookrightarrow U_\alpha$ of finite presentation [Ryd16]. For sufficiently large $\beta$, the immersion $Z_{\alpha \beta} \hookrightarrow U_\alpha \hookrightarrow U$ is affine, since the limit $Z_\alpha \hookrightarrow U_\alpha \hookrightarrow U$ is affine [Ryd15 Thm. C]. Thus $Z_{\alpha \beta} = W_\lambda$ for some $\lambda = \lambda(\alpha, \beta)$ for every $\alpha$ and every sufficiently large $\beta$. It follows that

$$\mathcal{G}_x \hookrightarrow \bigcap_{\lambda \in \Lambda} W_\lambda \hookrightarrow \bigcap_{\alpha} Z_\alpha = \mathcal{G}_x$$

and the result follows. \qed

3. Nisnevich Dévissage

In this section, we consider Nisnevich coverings for quasi-separated algebraic stacks. For schemes, this goes back to the work of [Nis89] with the most famous applications due to [MY99]. In the setting of equivariant schemes this was considered in [HKO15] §2. It was also considered for Deligne–Mumford stacks in [KO12] §§7-8. The restriction to quasi-separated algebraic stacks is so that we can give an intuitive definition in terms of residual gerbes.

Definition 3.1. A morphism of quasi-separated algebraic stacks $p: W \to X$ is a Nisnevich covering if it is étale and for every $x \in |X|$, there exists an $w \in |W|$ such that $p(w) = x$ and the induced map of residual gerbes $\mathcal{G}_w \to \mathcal{G}_x$ is an isomorphism.

Nisnevich coverings are stable under composition and base change.

Example 3.2. Let $X$ be a quasi-compact and quasi-separated scheme. Then there exists an affine scheme $W$ and a Nisnevich covering $p: W \to X$. Indeed, taking $W = \coprod_{i=1}^n U_i$, where the $\{U_i\}$ form a finite affine open covering of $X$ gives the claim. More generally, this holds for quasi-compact and quasi-separated algebraic spaces [RG71 Prop. 5.7.6].

Let $p: W \to X$ be a morphism of algebraic stacks. Recall that when $p$ is not representable, then a section of $p$ need not be a monomorphism. A monomorphic splitting sequence for $p$ is a sequence of quasi-compact open immersions

$$\emptyset = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X$$

such that $p$ restricted to $X_i \setminus X_{i-1}$, when given the induced reduced structure, admits a monomorphic section for each $i = 1, \ldots, r$. In this situation, we say that $p$ has a monomorphic splitting sequence of length $r$. 
We have the following characterization of Nisnevich coverings, which is well-known for noetherian schemes [MV99 Lem. 3.1.5].

**Proposition 3.3.** Let $X$ be a quasi-compact and quasi-separated algebraic stack and let $p: W \rightarrow X$ be a quasi-separated étale morphism. Then $p$ is a Nisnevich covering if and only if there exists a monomorphic splitting sequence for $p$.

**Proof.** Let $x \in |X|$ be a point. Then there exists an immersion $Z_x \hookrightarrow X$ of finite presentation, such that $x \in |Z_x|$, and a monomorphic section of $p|_{Z_x}$. Indeed, there is a monomorphic section of $p|_{Z_x}$ which extends to a monomorphic section of $p|_{Z_x}$ by Lemma [2.3] and [Ryd15 Prop. B.2 (i) and B.3 (iii)].

The $Z_x$ are constructible and we can thus cover $X$ by a finite number of the $Z_x$’s. We can thus filter $X$ by a sequence of quasi-compact open substacks $X_i$ such that $X_i \setminus X_{i-1}$ is contained in some $Z_x$. That is, we have obtained a monomorphic splitting sequence.

The following lemma outlines the key benefits of the Nisnevich topology: it is generated by particularly simple coverings (cf. [MV99 Prop. 1.4]).

**Lemma 3.4 (Nisnevich dévissage).** Let $S$ be a quasi-compact and quasi-separated algebraic stack and let $E \subseteq \text{Stack}_{\text{fp, ét}/S}$ be a full 2-subcategory containing all open immersions and closed under fiber products (e.g., one of the categories listed in the introduction). Let $D \subseteq E$ be a full 2-subcategory such that

- (N1) if $(X' \rightarrow X) \in E$ is an open immersion and $X \in D$, then $X' \in D$; and
- (N2) if $(U \twoheadrightarrow X), (X' \hookrightarrow X) \in E$, where $i$ is an open immersion and $f$ is an isomorphism over $X \setminus U$, then $X \in D$ whenever $U, X' \in D$.

If $p: W \rightarrow X$ is a Nisnevich covering in $E$ and $W \in D$, then $X \in D$.

**Proof.** By Proposition 3.3, there is a sequence of quasi-compact open immersions:

$$\emptyset = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X,$$

such that $f$ restricted to $X_i \setminus X_{i-1}$, when given the induced reduced structure, admits a monomorphic section for $i = 1, \ldots, r$. We will prove the result by induction on $r \geq 0$. If $r = 0$, then the result is trivial.

If $r > 0$, let $U = X_{r-1}$; then $U$ admits a splitting sequence of length $r-1$. By the inductive hypothesis and (N1), we may thus assume that $U \in D$. If $Z = (X \setminus U)_{\text{red}}$, then the restriction of $p$ to $Z$ admits a section $s$, which is a quasi-compact open immersion. It follows that $X' = p^{-1}(U) \cup s(Z) = W \setminus (p^{-1}(Z) \setminus s(Z))$ is a quasi-compact open subset of $W$. Let $f: X' \rightarrow X$ be the induced morphism; then $X' \in D$ and $f$ is an isomorphism over $X \setminus U$. By (N2), the result follows.

4. **Presentations of algebraic stacks with finite stabilizers**

The following theorem removes the separated diagonal assumption from [Hal16 Thm. B.5]. It will be crucial for the proofs of Theorems 3.1 and 5.1.

**Theorem 4.1.** Let $X$ be a quasi-compact and quasi-separated algebraic stack with quasi-finite diagonal. Then there exist morphisms of algebraic stacks

$$V \xrightarrow{v} W \xrightarrow{p} X$$

such that

- $V$ is an affine scheme;
- $v$ is finite, faithfully flat and of finite presentation; and
- $p$ is a Nisnevich covering of finite presentation with separated diagonal.

In addition,
(1) if $X$ has separated diagonal, then it can be arranged that $p$ is representable and separated; and

(2) if $X$ is Deligne–Mumford, then it can be arranged that $v$ is étale.

Proof. The proof is similar to [Ryd13, Prop. 6.11], [Ryd11a, Thms. 6.3 & 7.2] and [Hal16, Thm. B.5].

By [Ryd11a, Thm. 7.1], there is an affine scheme $U$ and a representable, quasi-finite, faithfully flat and finitely presented morphism $u: U \to X$. The Hilbert stack $H^\text{et}_{U/X} = \coprod_{d \geq 0} H^d_{U/X} \to X$ parametrizing quasi-finite representable morphisms to $U$ is algebraic and has quasi-affine—in particular, separated—diagonal [Ryd11b, Thm. 4.4]. Let $p: W = H^\text{et}_{U/X} \to X$ be the open substack of the Hilbert stack that parameterizes representable étale morphisms to $U$. Since $u$ is flat, it is readily seen that $p: W \to X$ is étale.

We now prove that $p$ is a Nisnevich covering. Let $x \in |X|$ be a point with residual gerbe $\mathcal{G}_x$. The restriction $u_x: U_x \to G_x$ is finite and flat. Thus, the identity $U_x \to U_x$ corresponds to a section $\mathcal{G}_x \to W$. It is readily seen that this is a monomorphic section (e.g., by considering the open substack $H \subseteq W$ below).

After replacing $W$ by a quasi-compact open subset containing the sections of a monomorphic splitting sequence (Proposition 3.3), we obtain a finitely presented Nisnevich covering $p: W \to X$. Let $v: V \to W$ be the universal family, which is finite (even étale if $u$ is étale), flat and of finite presentation. Then there is a 2-commutative diagram

$$
\begin{array}{ccc}
V & \xrightarrow{q} & U \\
\downarrow{v} & & \downarrow{u} \\
W & \xrightarrow{p} & X
\end{array}
$$

where $p$ and $q$ are étale. After shrinking $W$, we may assume that $v$ is surjective. Although $p$ and $q$ need neither be representable nor separated, we saw that $p$, and hence $q$, have separated diagonals. It follows that $V$ has separated diagonal, and hence so has $W$ [Ryd11a, Lem. A.4]. We may replace $X$ by $W$ and assume that $X$ has separated diagonal.

When $X$ has separated diagonal, the presentation $u$ is separated. Consider the substack $H = \text{Hilb}^\text{open}_{U/X} \subseteq W$ parameterizing open and closed immersions into $U$ over $X$. In general $H$ is not algebraic but since $u$ is separated it is an open substack of $W$ and $H \to S$ is representable and separated [Ryd11b, Thm. 4.1]. We may thus replace $W$ with a quasi-compact open subset of $H$ containing the sections. Then we obtain a commutative diagram as above where $p$ and $q$ are étale, representable and separated. By Zariski’s Main Theorem [LMB, Thm. A.2], $q$ is quasi-affine. By [Ryd13, Thm. 5.3], $W$ has a coarse space $\pi: W \to W_{cs}$ such that $W_{cs}$ is a quasi-affine scheme and $\pi \circ v$ is affine (and integral). By Example 3.2 we may further reduce to the situation where $W_{cs}$ is an affine scheme. Then $V$ is affine and the result follows. □

Remark 4.2. A special case of (1) is when $X$ has finite inertia. Then one can give an alternative proof of Theorem 4.1 using that $X$ admits a coarse space $X \to X_{cs}$, and that Nisnevich-locally on $X_{cs}$, we can find a finite flat presentation of $X$. Indeed, one immediately reduces to the case where $X_{cs}$ is local henselian and then a quasi-finite flat presentation $U \to X$ splits as $U = V \amalg V'$ where $V \to X$ is finite and surjective.
Let $f : X \to S$ be a morphism of algebraic stacks. Let $HS_{X/S}$ be the Hilbert stack of $f$. The Hilbert stack of $f$ parameterizes quasi-finite and representable morphisms to $X$ that are proper over the base. In [HR15b, HR14], it was proved that $HS_{X/S}$ was algebraic when $f$ has quasi-finite and separated diagonal. The proof of this relies on the results of [HR14], whose methods are quite involved and may not be so familiar to the reader.

In this article, we will only need a small piece of $HS_{X/S}$: the open substack $HS^{qfb}_{X/S}$ consisting of those families that are quasi-finite (though not necessarily representable) over the base. We will call this the Hilbert stack of stacky points. Using Nisnevich coverings, we will be able to deduce the algebraicity of the Hilbert stack of stacky points from the well-known algebraicity result in the case where $f$ is separated, which is much easier (e.g. [Lie06], [Hal17, Thm. 9.1] and [HR15b, Thm. A(i)]).

**Theorem 5.1.** If $f : X \to S$ is a morphism of algebraic stacks with quasi-compact and separated diagonal, then $HS^{qfb}_{X/S}$ is an algebraic stack with quasi-affine diagonal over $S$. If $f$ is locally of finite presentation (resp. is separated), then $HS^{qfb}_{X/S}$ is locally of finite presentation (resp. has affine diagonal).

To prove Theorem 5.1 we first prove a result on Weil restrictions.

**Proposition 5.2.** Let $Z \to S$ be a quasi-finite, proper and flat morphism of finite presentation between quasi-separated algebraic stacks. If $U \to Z$ is a quasi-separated morphism with quasi-finite diagonal, then the Weil restriction $R_{Z/S}(U) \to S$ is a quasi-separated algebraic stack. Moreover, if $U \to Z$ has separated diagonal, then $R_{Z/S}(U) \to S$ has quasi-affine diagonal.

If $U \to Z$ has separated diagonal, it can be deduced that $R_{Z/S}(U)$ is algebraic with quasi-affine diagonal using [HR15b, Thm. 2.3(vi)]. This relies on [HR14], however. We will avoid the reliance on [HR14] and the separated diagonal assumption when $Z \to S$ is quasi-finite using a simple bootstrapping process and Theorem 4.1.

**Proof of Proposition 5.2.** A standard argument shows that properties (2), (3), and (4) are preserved by taking Weil restrictions whenever the Weil restrictions in question exist, cf. [HR15b, Rem. 2.5]. To prove (1) when $R_{Z/S}(U) \to S$ is already known to be a quasi-separated algebraic stack, we may replace $S$ with a residual gerbe $G_s$ for some point $s \in |S|$. Then $|Z|$ is finite and discrete. Thus, if $U \to Z$ is a Nisnevich covering, then $U \to Z$ has a monomorphic section. It follows that there is a monomorphic section $S \to R_{Z/S}(U)$.

We make the following well-known observation: if $u : U_1 \to U_2$ is a morphism of algebraic stacks over $Z$, then the base change of $R_{Z/S}(U) : R_{Z/S}(U_1) \to R_{Z/S}(U_2)$ along a morphism $T \to R_{Z/S}(U_2)$, corresponding to a $Z$-morphism $Z \times_S T \to U_2$, is isomorphic to $R_{Z \times_Z T/T}((Z \times_Z T) \times_{U_2} U_1)$. It follows that if $P$ is a property of morphisms of algebraic stacks that is smooth-local on the target, then $R_{Z/S}(U)$ is $P$ if $R_{Z/S}(U) \to S$ is $P$ for all affine $S$ and all $U \to Z$ satisfying $P$. 


We next address the algebraicity. If \( U \to Z \) is separated (resp. separated and representable), then \( R_{Z/S}(U) \to S \) is well-known to be algebraic with affine diagonal (resp. representable and separated), see [HR15b, Thm. 2.3(v)]).

The algebraicity is smooth local on \( S \), so we may assume that \( S \) is an affine scheme. Every section of \( U \to Z \) factors through a quasi-compact open subset and Weil-restrictions of open substacks are open substacks, hence we may assume that \( U \) is quasi-compact. Theorem 4.1 implies that there is a Nisnevich covering \( p: W \to U \) such that \( W \) has finite diagonal and \( W \to U \) has separated diagonal. By the case already considered, \( R_{Z/S}(W) \to S \) is algebraic with affine diagonal. Consider the induced morphism \( R_{Z/S}(p): R_{Z/S}(W) \to R_{Z/S}(U) \).

If \( U \to Z \) has separated diagonal, then Theorem 4.1 even says that we can choose the Nisnevich covering \( p: W \to U \) to be separated and representable. The separated case already considered and \( [\text{iv}] \) now establishes that \( R_{Z/S}(p) \) is a representable and separated Nisnevich covering. Hence, \( R_{Z/S}(U) \to S \) is algebraic. To see that it has quasi-affine diagonal, we note that \( R_{Z/S}(U) \times_S R_{Z/S}(U) \cong R_{Z/S}(U \times_Z U) \). In particular, \( \Delta_{R_{Z/S}(U)} \cong R_{Z/S}(\Delta_{U/Z}) \). Since \( \Delta_{U/Z} \) is quasi-affine, \( R_{Z/S}(\Delta_{U/Z}) \) is quasi-affine [HR15b, Thm. 2.3(iii)].

If \( U \to Z \) does not have separated diagonal, then \( p: W \to U \) still has separated diagonal. Hence, by the cases already considered, \( R_{Z/S}(p) \) is algebraic and a Nisnevich étale covering. It follows that \( R_{Z/S}(U) \) is algebraic, but we still need to prove that it is quasi-separated. Repeating the argument above on separation conditions for \( R_{Z/S}(U) \to S \), the quasi-separatedness follows from \( [\text{v}] \).

It remains to show \( [\text{v}] \): the Weil restriction \( R := R_{Z/S}(U) \to S \) is quasi-compact if \( U \to Z \) is quasi-compact. This claim is smooth local on \( S \) so we may assume that \( S \) is affine. Pick a quasi-finite flat presentation \( Z' \to Z \) and let \( Z'' = Z' \times_Z Z' \) and \( Z''' = Z' \times_Z Z' \times_Z Z' \). To show that \( R \) is quasi-compact, we may replace \( S \) with a stratification. We may thus assume that \( Z' \to S \) is finite. Then \( R' := R_{Z'/S}(U \times_Z Z') \to S \), \( R'' := R_{Z''/S}(U \times_Z Z'') \to S \) and \( R''' := R_{Z'''/S}(U \times_Z Z''') \to S \) are quasi-compact and quasi-separated algebraic stacks [Ryd11b, Prop. 3.8 (xiii) & (xix)]. If we define \( P \) (descent data without the descent condition) by the cartesian square

\[
\begin{array}{ccc}
R' & \xrightarrow{(\pi_1, \pi_2)} & P \\
\downarrow & & \downarrow \\
R'' \times_S R'' & \xrightarrow{\Delta} & R''
\end{array}
\]

then there is a cartesian square

\[
\begin{array}{ccc}
P & \xleftarrow{\tau} & R \\
\downarrow & & \downarrow \\
I_{R'''} & \xleftarrow{e} & R'''
\end{array}
\]

by fppf descent [Ols07, Rmk. 4.4]. It follows that \( R \) is quasi-compact. \( \square \)

We can now prove Theorem 5.1.

**Proof of Theorem 5.1.** We may assume that \( S \) is an affine scheme. If \( X^qf \subseteq X \) denotes the open substack where \( X \) has a quasi-finite diagonal, then it is clear that \( \text{HS}^qf_{X/S} = \text{HS}^qf_{X/S} \). Thus we may assume that \( X \) has quasi-finite and separated diagonal. Further standard reductions permit us to assume that \( X \) is also quasi-compact. By Theorem 4.1 there is a finitely presented, representable, and separated Nisnevich covering \( p: W \to X \) such that \( W \) admits a finite flat and finitely presented covering by an affine scheme \( V \). If \( X \) is separated, we instead let \( W = X \). In either
case, \( W \) has finite diagonal. By [HR15b, Thm. A(i)], \( \mathbb{HS}^q_W/S \) is an algebraic stack with affine diagonal.

Let \( T \) be an affine scheme and let \( (\mathcal{Z} \to X \times S T) \in \mathbb{HS}^q_W/S(T) \). It is well-known that the following diagram is 2-cartesian:

\[
\begin{array}{ccc}
R_{\mathcal{Z}/T}((W \times S T) \times_{X \times S} \mathcal{Z}) & \to & T \\
\downarrow & & \downarrow \\
\mathbb{HS}^q_W/S & \to & \mathbb{HS}^q_{X/S},
\end{array}
\]

and we conclude that \( \mathbb{HS}^q_W/S \to \mathbb{HS}^q_{X/S} \) is a finitely presented, representable, and separated Nisnevich covering (Proposition 5.2). The theorem follows. \( \square \)

**Example 5.3.** Theorem 5.1 is false if \( X \to S \) has non-separated diagonal. This is similar to the main result of [LS08] (cf. [HR14]). For an explicit example, consider \( S = A_1^1 \), where \( k \) is a field, and let \( G = (\mathbb{Z}/2\mathbb{Z})_S \). Let \( H \subseteq G \) be the étale subgroup scheme which is the complement of the non-trivial element lying over the origin in \( S \). The quotient \( G/H \) is non-separated (it is just the line with the doubled origin). Let \( X = B_S(G/H) \). Let \( S_n = \text{Spec}(k[x]/x^{n+1}) \) and \( \hat{S} = \text{Spec } k[[x]] \). The natural map \((B_SG) \times_S S_n \to X \times_S \hat{S}_n\) is representable (even an isomorphism), but there is no extension of this to a representable morphism \( Y \to X \times_S \hat{S} \), where \( Y \to \hat{S} \) is proper and flat.

**Remark 5.4.** If \( X \to S \) is non-separated, then the natural object to consider is the 2-stack parameterizing not necessarily representable morphisms \( Z \to X \) that are quasi-finite and flat over the base. This 2-stack ends up being algebraic because the proof of Theorem 5.1 holds verbatim. If \( X \to S \) is flat and we restrict to the 2-substack parameterizing those \( Z \to X \) that are also étale, then this is an étale 2-stack. In particular, it is an étale 2-gerbe over a 1-stack. Unfortunately, this 1-stack does not carry a universal family, which makes applying the result difficult. In particular, to prove dévissage results for morphisms with non-separated diagonals, it appears necessary to enter the world of higher stacks, cf. Remark 6.2.

### 6. Non-representable presentations

The following theorem combines and extends [Ryd13, Prop. 6.11] and [Ryd11a, Thm. 6.3]. It makes crucial use of Theorem 5.1.

**Theorem 6.1.** Let \( X \) be a quasi-compact and quasi-separated algebraic stack and let \( u: U \to X \) be a quasi-finite and faithfully flat morphism of finite presentation with separated diagonal. Then there exists a commutative diagram of algebraic stacks

\[
\begin{array}{ccc}
V & \to & U \\
\downarrow & & \downarrow \uparrow \\
W & \to & X
\end{array}
\]

such that

- \( v \) is quasi-finite, proper and faithfully flat of finite presentation;
- \( p \) is a Nisnevich étale covering of finite presentation with separated diagonal; and
- \( q \) is an étale morphism of finite presentation with separated diagonal.

In addition,

(1) if \( u \) is representable, then it can be arranged that \( v \) is representable;
(2) if $u$ is separated, then it can be arranged that $p$ and $q$ are separated and representable; and
(3) if $u$ is étale, then it can be arranged that $v$ is étale.

Proof. Argue exactly as in the proof of the first part of Theorem 4.1. As before we take $W = \text{Hilb}_{U/X}$, the open substack of the Hilbert stack $\text{Hilb}_{U/X}$ parameterizing étale morphisms to $U$. Since $U \to X$ is quasi-finite, $\text{Hilb}_{U/X} = \text{Hilb}_{U/X}^{qfb}$ is algebraic with quasi-affine diagonal (Theorem 5.1). As before, it follows that $W \to X$ is étale with quasi-affine, hence separated, diagonal. If $u$ is separated, we replace $W$ with the open substack $\text{Hilb}_{U/X}^{\text{open}}$ which is separated and representable over $X$. □

Remark 6.2. If $u$ does not have separated diagonal in Theorem 6.1 then using the Hilbert 2-stack of Remark 5.3 we would arrive at the conclusion of the Theorem except that $p$ and $q$ need not have separated diagonals and are merely 2-representable, though $v$ is still 1-representable. Here $n$-representable means represented by algebraic $n$-stacks. In particular, $V$ and $W$ are algebraic 2-stacks.

7. Applications

In this section, we use non-representable étale dévissage to relax some separatedness conditions in the approximation results of [Ryd15] and the compact generation results of [HR17].

Lemma 7.1. Let $S$ be a quasi-compact and quasi-separated algebraic stack. Let $X$ be a quasi-compact and quasi-separated algebraic stack over $S$ and let $\pi: \mathcal{X} \to X$ be a proper fppf gerbe. Suppose $\mathcal{X} = \varprojlim_{\lambda \in \Lambda} \mathcal{X}_{\lambda}$ where $\mathcal{X}_{\lambda}$ are algebraic stacks of finite presentation over $S$ and $g_{\lambda}: \mathcal{X}_{\lambda} \to \mathcal{X}_{\lambda}$ are affine morphisms. Then for all sufficiently large $\lambda$, there is a commutative diagram

\begin{equation}
\begin{array}{ccc}
\mathcal{X} & \overset{\mathcal{g}_{\lambda}}{\longrightarrow} & \mathcal{X}_{\lambda} \\
\pi & \mathcal{X}_{\lambda} \downarrow & \mathcal{X}_{\lambda} \\
X & \overset{\pi_{\lambda}}{\longrightarrow} & X_{\lambda}
\end{array}
\end{equation}

where $i_{\lambda}$ is a finitely presented closed immersion, $\pi_{\lambda}$ is a proper fppf gerbe and the square is cartesian. In particular, $X \to X_{\lambda}$ is affine and $X_{\lambda} \to S$ is of finite presentation.

Proof. The map $\pi$ gives an exact sequence of group objects over $\mathcal{X}$

$$0 \to I_{\mathcal{X}/X} \to I_{\mathcal{X}/S} \to \pi^{*}I_{X/S}.$$ 

That $\pi$ is an fppf gerbe of finite presentation implies that $I_{\mathcal{X}/X}$ is flat and of finite presentation. Conversely, given a flat subgroup $G \subseteq I_{\mathcal{X}/S}$ of finite presentation, there exists a rigidification: an algebraic stack $\mathcal{X} \sslash G$ over $S$ together with an fppf gerbe $\mathcal{X} \to \mathcal{X} \sslash G$ of finite presentation such that the relative inertia is $G$ [AOV08 Thm. A.1].

Let $G = I_{\mathcal{X}/X}$ and fix an index $\alpha \in \Lambda$. The inertia stack of $I_{\mathcal{X}/S}$ does not pull-back to $I_{\mathcal{X}/S}$ but the canonical map $I_{\mathcal{X}/S} \to I_{\mathcal{X}/S} \times_{\mathcal{X}} \mathcal{X}$ is a closed subgroup stack. Since $G \to \mathcal{X}$ and $I_{\mathcal{X}/S} \to I_{\mathcal{X}/S}$ are of finite presentation, there is, by standard approximation methods [Ryd15 Props. B.2, B.3], an index $\lambda \geq \alpha$ and a subgroup $G_{\lambda} \subseteq I_{\mathcal{X}/S} \times_{\mathcal{X}} \mathcal{X}_{\lambda}$ of finite presentation that pulls back to $G \leftarrow I_{\mathcal{X}/S} \times_{\mathcal{X}} \mathcal{X}_{\lambda}$. After increasing $\lambda$, we may assume that $G_{\lambda} \to \mathcal{X}_{\lambda}$ is flat and proper [Ryd15 Prop. B.3].

We now address the problem that $G_{\lambda}$ need not be a subgroup of $I_{\mathcal{X}/S}$. Let $H_{\lambda} = G_{\lambda} \cap I_{\mathcal{X}/S}$ as subgroups of $I_{\mathcal{X}/S} \times_{\mathcal{X}} \mathcal{X}_{\lambda}$. Then $H_{\lambda} \to G_{\lambda}$ is a finitely
presented closed subgroup and \( H_\lambda \times_{X_\lambda} \mathcal{X} \to G_\lambda \times_{X_\lambda} \mathcal{X} \) is an isomorphism. It follows that the Weil restriction \( \mathcal{X}_\lambda^c := R_{G_\lambda/X_\lambda}(H_\lambda) \) is a finitely presented closed substack of \( \mathcal{X}_\lambda \) and that \( g_\lambda: \mathcal{X} \to \mathcal{X}_\lambda \) factors uniquely through \( \mathcal{X}_\lambda^c \). Also note that after restricting to \( \mathcal{X}_\lambda^c \), the closed subgroup \( H_\lambda \to G_\lambda \) becomes an isomorphism.

We thus have the subgroup \( G_\lambda^c := G_\lambda|_{\mathcal{X}_\lambda^c} \to I_{\mathcal{X}_\lambda^c} \) which is proper and flat over \( \mathcal{X}_\lambda^c \).

Let \( X_\lambda^c = \mathcal{X}_\lambda^c \sslash G_\lambda^c \). It remains to prove that we have a cartesian diagram. Since \( \mathcal{X} \to X \) is initial among maps \( \mathcal{X} \to Y \) such that \( G \to I_{\mathcal{X}/S} \) factors through \( I_{\mathcal{X}/Y} \to I_{\mathcal{X}/S} \), we have a map \( X \to X_\lambda^c \). This induces a map between gerbes \( \mathcal{X} \to \mathcal{X}_\lambda^c \times_{X_\lambda^c} X \) over \( X \). This is a stabilizer-preserving morphism, i.e., \( I_{\mathcal{X}/X} = G \to I_{\mathcal{X}_\lambda^c/X_\lambda^c} \times_{X_\lambda^c} \mathcal{X} = G_\lambda^c \times_{X_\lambda^c} \mathcal{X} \) is an isomorphism. But a stabilizer-preserving morphism between gerbes is an isomorphism.

We can now remove most of the representability assumption in [Ryd15] Lemma 7.9.

**Proposition 7.2.** Let \( S \) be a pseudo-noetherian stack and let \( X \to S \) be a morphism of algebraic stacks. Let \( W \to X \) be an étale surjective morphism of finite presentation with separated diagonal (e.g., representable). If \( W \to S \) can be approximated, then so can \( X \to S \).

**Proof.** We will apply étale dévissage (Theorem D’). Let \( D \subseteq E = \text{Stack}_{\text{sep, fp, ét/S}} \) be the full subcategory of morphisms \( Y \to X \) such that \( Y \to S \) is of strict approximation type or, equivalently, has an approximation [Ryd15] Prop. 4.8. Then (D1) is satisfied by definition; (D2) for finite morphisms is [Ryd15] Prop. 2.12 (ii) and (D3) is [Ryd15] Lem. 7.8. It remains to prove (D2) for proper non-representable morphisms. Thus, let \( Y'' \to Y \) be a proper étale surjective morphism in \( E \). There is a canonical factorization \( Y'' \to Y' \to Y \) where the first morphism is an étale gerbe and the second is finite étale. It is thus enough to prove (D2) when \( Y'' \to Y \) is a proper étale gerbe.

By assumption \( Y'' \to S \) has an approximation and can thus be written as \( Y'' = \lim_{\lambda} Y'_{\lambda} \) where \( Y'_{\lambda} \to S \) are of finite presentation and \( Y'' \to Y'_{\lambda} \) is affine for every \( \lambda \).

By Lemma 7.1 we have a cartesian diagram

\[
\begin{array}{ccc}
Y'' & \longrightarrow & Y'^{\circ} \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Y_{\lambda}^{\circ}.
\end{array}
\]

of algebraic stacks over \( S \) where \( Y \to Y_{\lambda}^{\circ} \) is affine and \( Y_{\lambda}^{\circ} \to S \) is of finite presentation. Thus, \( Y \to S \) has an approximation.

In [Ryd15] it is shown that quasi-compact algebraic stacks with quasi-finite and locally separated diagonal can be approximated and are pseudo-noetherian. We can now remove the locally separatedness assumption.

**Corollary 7.3.** Let \( X \) be a quasi-compact algebraic stack with quasi-finite and quasi-separated diagonal. Then \( X \to \text{Spec} \mathbb{Z} \) has an approximation. In particular, \( X \) is pseudo-noetherian.

**Proof.** By Theorem 4.4 there is an étale surjective morphism \( W \to X \) of finite presentation with separated diagonal (a Nisnevich cover) and a finite faithfully flat morphism \( V \to W \) of finite presentation where \( V \) is an affine scheme. We conclude that \( W \) has an approximation by [Ryd15] Prop. 2.12 (ii)] and that \( X \) has an approximation by Proposition 7.2.

We can also establish the following improvement of [HR17] Thm. A] in equicharacteristic 0, where it was proved for stacks with quasi-finite and separated diagonal.
Theorem 7.4. Let $X$ be a quasi-compact and quasi-separated Deligne–Mumford stack of equicharacteristic 0. Then the unbounded derived category $\mathcal{D}_{qc}(X)$, of $\mathcal{O}_X$-modules with quasi-coherent cohomology, is compactly generated by a single perfect complex. Moreover, for every quasi-compact open subset $U \subseteq X$, there exists a compact perfect complex with support exactly $X \setminus U$.

Proof. We apply Theorem 1; let $\mathcal{D} \subseteq \mathcal{E} = \text{Stack}_{\text{sep, fp, ét}/X}$ be the full subcategory consisting of those morphisms of Deligne–Mumford stacks $(W \to X)$, where for every quasi-compact open immersion $V \subseteq W$ we have that $V$ satisfies the conclusion of the Theorem. This makes condition (I1) a triviality. Condition (I2) follows immediately from [HR17, Thm. A]. For Condition (I3) we use the theory developed in [HR17] §§5-6, with the following minor changes. In [HR17] Ex. 5.2, the working example throughout those sections, they take $\mathcal{D}$ to consist of representable and finitely presented morphisms to $X$; we will take $\mathcal{D} = \mathcal{E}$. The main difference is that $\mathcal{D}$ is now a 2-category, but the results go through without change. Since all morphisms of Deligne–Mumford stacks in equicharacteristic 0 are concentrated (combine [HR17, Lem. 2.5(2)] with [HR15a, Thm. C]), the resulting $(\mathcal{L}, \mathcal{D})$-presheaf of triangulated categories is admissible in the sense of [HR17] Defn. 6.1]; also see [HR17] Ex. 6.2 for further details and notations. Condition (I3) now follows from [HR17] Prop. 6.8. □

Corollary 7.5. If $X$ is a noetherian Deligne–Mumford stack of equicharacteristic 0, then there is an equivalence of categories:

$$\mathcal{D}(\text{QCoh}(X)) \to \mathcal{D}_{qc}(X).$$


Remark 7.6. If $p: W \to X$ is a morphism of algebraic stacks and $W$ has separated diagonal, then $p$ has separated diagonal. This means that the étale presentations appearing in [AHR15, AHR14] always have separated diagonal.

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