GAGA THEOREMS

JACK HALL

Abstract. We prove a new and unified GAGA theorem. This recovers all analytic and formal GAGA results in the literature, and is also valid in the relative and non-noetherian setting. Our method can also be used to establish various Lefschetz theorems.

1. Introduction

Let X be a noetherian scheme. Frequently associated to X is a flat morphism of locally ringed spaces:

$$c\colon \mathcal{X} \to X,$$

where $\mathcal{X}$ is some type of analytic space with coherent structure sheaf. When X is proper over $\operatorname{Spec} R$, where R is a suitable noetherian ring (usually connected to the construction of the analytic object $\mathcal{X}$), there is frequently an induced comparison isomorphism on cohomology of coherent sheaves:

$$\mathrm{H}^i(X, F) \simeq \mathrm{H}^i(\mathcal{X}, c^*F)$$

and an equivalence of abelian categories of coherent sheaves:

$$c^*\colon \mathsf{Coh}(X) \simeq \mathsf{Coh}(\mathcal{X}).$$

Since Serre's famous paper [GAGA], such results have been called "GAGA theorems". Restricting these comparison isomorphisms and equivalences to specific subcategories of sheaves (e.g., vector bundles or finite étale algebras) leads to both local and global "Lefschetz Theorems" [SGA2]. We briefly recall some examples of these phenomena below.

1.1. Archimedean analytification.

Assume X is locally of finite type over $\operatorname{Spec} \mathbb{C}$. Naturally associated to X is an analytic space X_{an}. This consists of endowing the $\mathbb{C}$-points of X with its Euclidean topology and its sheaf of holomorphic functions. This is the setting of the original GAGA theorem proved by Serre [GAGA]. This was all generalized to proper schemes over $\operatorname{Spec} \mathbb{C}$ in [SGA1, XII.4.4] and to proper Deligne–Mumford stacks in Toën's thesis [Toë99, 5.10]. Generalizations have recently been developed for higher derived stacks by Porta–Yu [PY16]. There is also a relative version for projective morphisms, due to Abramovich–Temkin [AT18, App. C], which we generalize (Example 9.8). Also if X is a scheme, proper over $\operatorname{Spec} \mathbb{R}$, then there is a notion of a real analytification and suitable GAGA results due to Huisman [Hui02].

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1.2. **Formal completion.** Let $Z \subseteq X$ be a closed subscheme. Then we may form the formal completion $\hat{X}_{/Z}$ of X along Z. The locally ringed space $\hat{X}_{/Z}$ has underlying topological space Z and sheaf of functions the formal power series along Z; that is, $\mathcal{O}_{\hat{X}_{/Z}} = \varprojlim_n \mathcal{O}_X/I^n$, where $Z = V(I)$. There are recent variations of this in the non-noetherian situation due to Fujiwara–Kato [FK13] and the Stacks Project [Stacks]. This is also the typical setting for local and global Lefschetz theorems, which have seen recent interest [Kol13, BJ14, Kol16].

1.3. **Non-archimedean analytification.** Assume X is separated and locally of finite type over $\operatorname{Spec} K$, where K is a complete non-archimedean field. Then there are various analytifications associated to X: the Berkovich X_{Berk}, the adic X_{adic} [Hub94], and the rigid X_{rig}. These constructions are all quite different, but all have equivalent topoi. There is also a GAGA theorem in this context, due to Köpf [Köp74]. Also, see the work of Conrad [Con06] for a more recent account. In special situations, there are even analytifications when X is not locally of finite type—this is the setting of the Fargues–Fontaine curve [FF18].

1.4. **Unification.** All of these results have previously been proved separately, though their general strategies are very similar. Indeed, once the cohomology of line bundles on projective space is completely determined (Cartan's Theorems A & B), the results are proved directly for projective morphisms. Using Chow's Lemma, a dévissage argument is then used to reduce the case of a proper morphism to the projective situation.

The main theorem of this article is that these GAGA results are true much more generally and can be put into a single framework. There is no dévissage to the projective situation and all existing results follow very easily from ours (see §9). We state one such result in the noetherian situation for locally G-ringed spaces.

Theorem A. *Let R be a noetherian ring. Let X be a proper scheme over $\operatorname{Spec} R$. Let $c\colon \mathcal{X} \to X$ be a morphism of locally G-ringed spaces. Let X_{cl} be the set of closed points of X and let $\mathcal{X}_{\mathrm{cl},c} = c^{-1}(X_{\mathrm{cl}})$. Assume that*

(1) *$\mathcal{O}_{\mathcal{X}}$ is coherent;*
(2) *if $\mathcal{F} \in \mathsf{Coh}(\mathcal{X})$, then $\oplus_i \mathrm{H}^i(\mathcal{X}, \mathcal{F})$ is a finitely generated R-module;*
(3) *$c\colon \mathcal{X}_{\mathrm{cl},c} \to X_{\mathrm{cl}}$ is bijective;*
(4) *if $\mathcal{F} \in \mathsf{Coh}(\mathcal{X})$ and $\mathcal{F}_x = 0$ for all $x \in \mathcal{X}_{\mathrm{cl},c}$, then $\mathcal{F} = 0$; and*
(5) *if $x \in \mathcal{X}_{\mathrm{cl},c}$, then $\mathcal{O}_{X,c(x)} \to \mathcal{O}_{\mathcal{X},x}$ is flat and $\kappa(c(x)) \to \kappa(c(x)) \otimes_{\mathcal{O}_{X,c(x)}} \mathcal{O}_{\mathcal{X},x}$ is an isomorphism.*

Then the comparison map:

$$\mathrm{H}^i(X, F) \to \mathrm{H}^i(\mathcal{X}, c^*F)$$

is an isomorphism for all coherent sheaves F on X and

$$c^*\colon \mathsf{Coh}(X) \to \mathsf{Coh}(\mathcal{X})$$

is an exact equivalence of abelian categories.

Our method is very powerful. Theorem A is a consequence of a general non-noetherian result (Theorem 8.1), where we do not even need to assume flatness or properness. In future work, we will address algebraic stacks and their derived counterparts. Our method derives from the innovative approach to the non-noetherian formal GAGA results proven in the Stacks Project [Stacks, Tag 0DIJ]. This result also follows from our Theorem 8.1 (see Example 9.7).

Remark 1.1. Basic properties of completions of noetherian local rings show that Theorem A(5) is implied by:

(5') if $x \in \mathcal{X}_{\mathrm{cl},c}$, then $\mathcal{O}_{\mathcal{X},x}$ is noetherian and the morphism $\mathcal{O}_{X,c(x)} \to \mathcal{O}_{\mathcal{X},x}$ induces an isomorphism on maximal-adic completions.

While it is possible to state a version of Theorem A for algebraic spaces, this requires a discussion of points in henselian topoi. While these have been discussed elsewhere [Con10], we felt that this would be a significant detour. We instead prove Theorem 9.1, which should be just as easy to apply as Theorem A in practice.

Applying Theorem A to a quasi-compact and separated morphism of schemes $c\colon Y \to X$, a short argument using Corollary 3.10 shows that c is an isomorphism. This is what we should expect from the Gabriel spectrum [Gab62]: every exact equivalence $F\colon \mathsf{Coh}(X) \to \mathsf{Coh}(Y)$ that sends $\mathcal{O}_X$ to $\mathcal{O}_Y$ is of the form c^* for a unique isomorphism of schemes $c\colon Y \to X$. We also wish to point out that the conditions of Theorem A are essentially necessary (see Remark 9.3).

In fact, our techniques are so malleable, we can even prove Lefschetz-style theorems as below.

Theorem B. *Let X be a quasi-compact and quasi-separated algebraic space. Let $c\colon \mathcal{X} \to X_{\text{ét}}$ be a morphism of ringed topoi and $i\colon Z \hookrightarrow X$ a closed immersion. Assume*

(1) *$X - Z$ is quasi-affine,*
(2) *$\mathcal{O}_X$ is coherent or $i_*\mathcal{O}_Z$ is perfect,*
(3) *c and i are tor-independent,*
(4) *$\mathsf{D}^-_{\mathrm{pc}}(X_{\text{ét}}, i_*\mathcal{O}_Z) \simeq \mathsf{D}^-_{\mathrm{pc}}(\mathcal{X}, c^*i_*\mathcal{O}_Z)$,*
(5) *$\mathrm{H}^0(X, \mathcal{O}_X) \simeq \mathrm{H}^0(\mathcal{X}, \mathcal{O}_\mathcal{X})$ and $\mathrm{H}^1(X, \mathcal{O}_X) \hookrightarrow \mathrm{H}^1(\mathcal{X}, \mathcal{O}_\mathcal{X})$.*

Then the comparison map:

$$\mathrm{H}^0(X, F) \to \mathrm{H}^0(\mathcal{X}, c^*F)$$

is an isomorphism for all vector bundles F on X and

$$c^*\colon \mathsf{Vect}(X) \to \mathsf{Vect}(\mathcal{X})$$

is fully faithful.

We will see in Theorem 2.1 that Theorem B implies the full faithfulness of the analytification functor for vector bundles on the Fargues–Fontaine curve [FF18, KL16]. Also see Theorem 9.9 for a variant of [BJ14, §1].

Theorem B is a special case of a more general result (Theorem 6.1) that does not even require tor-independence and also compares the higher cohomology groups. Essential surjectivity results along the lines of Theorem B are much more subtle. Certainly, some follow readily from our general GAGA result (Theorem 9.1). Lacking a good localization and support theory for analytic sheaves makes more general results difficult to come by (see Remark 2.2). There are of course well-established techniques to handle related problems—that continue to be developed to this day—using methods from commutative algebra.

The following technical question arose in this work. It would be nice to have a conclusive answer.

Question 1.2. *Let X be a quasi-compact and separated algebraic space. Let $i\colon Z \hookrightarrow X$ be a closed immersion. What conditions on X and Z guarantee the existence of* *a perfect complex P on X such that P lies in the thick closure of $\mathsf{D}^-_{\mathrm{pc}}(Z)$ in $\mathsf{D}_{\mathrm{qc}}(X)$ and has support $|Z|$?*

In Lemma 3.4, we prove that if $\mathcal{O}_X$ is coherent then there is always a P as in Question 1.2. This is sufficient for our intended applications. It is certainly necessary that the complement of Z in X is quasi-compact.

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2. Special cases

In this section we will establish Theorems A and B in some special cases, where certain optimizations can be made. We do this to advertise the simplicity of the core argument and illustrate the general strategy.

2.1. **GAGA.** Here we will prove Theorem A when R is assumed to be a field k and $X \to \operatorname{Spec} k$ is a smooth projective morphism. We will further assume that $\mathcal{X}$ is a locally ringed space and that c is flat and bijective on closed points.

We will be concerned with the triangulated categories $\mathsf{D}^b_{\mathsf{Coh}}(X)$ and $\mathsf{D}^b_{\mathsf{Coh}}(\mathcal{X})$. The objects of these categories are bounded complexes of $\mathcal{O}_X$-modules and $\mathcal{O}_{\mathcal{X}}$-modules, respectively, with coherent cohomology sheaves. Since the morphism $c\colon \mathcal{X} \to X$ is flat, there is a derived pullback functor

$$c^*\colon \mathsf{D}^b_{\mathsf{Coh}}(X) \to \mathsf{D}^b_{\mathsf{Coh}}(\mathcal{X}).$$

Our first observation is that condition (2) implies that this functor admits a right adjoint $c_*\colon \mathsf{D}^b_{\mathsf{Coh}}(\mathcal{X}) \to \mathsf{D}^b_{\mathsf{Coh}}(X)$. This is a non-trivial result, but it is a simple consequence of a theorem of Bondal and van den Bergh [BB03, Thm. 1.1] that we will now briefly explain. Let $\mathcal{M} \in \mathsf{D}^b_{\mathsf{Coh}}(\mathcal{X})$ and consider the cohomological functor

$$F_{\mathcal{M}}\colon \mathsf{D}^b_{\mathsf{Coh}}(X)^\circ \to \mathsf{Vect}(k)\colon P \mapsto \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(c^*P, \mathcal{M}).$$

Since X is smooth, its local rings are regular. In particular, if $P \in \mathsf{D}^b_{\mathsf{Coh}}(X)$, then P is a perfect complex. Hence, c^*P is a perfect complex and

$$F_{\mathcal{M}}(P) = \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(c^*P, \mathcal{M}) \simeq \operatorname{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}, c^*(P^\vee) \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{M}) \simeq \mathcal{H}^0(\mathsf{R}\Gamma(\mathcal{X}, c^*(P^\vee) \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{M})),$$

where $P^\vee = \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(P, \mathcal{O}_X)$ is the dual of P. Now $c^*(P^\vee)$ is also a perfect complex and so $c^*(P^\vee) \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{M} \in \mathsf{D}^b_{\mathsf{Coh}}(\mathcal{X})$. It now follows easily from condition (2) that $\oplus_i F_{\mathcal{M}}(P[i])$ is a finite-dimensional k-vector space. This means that the cohomological functor $F_{\mathcal{M}}$ is of *finite type* and [BB03, Thm. 1.1] implies that there is an $M \in \mathsf{D}^b_{\mathsf{Coh}}(X)$ such that $F_{\mathcal{M}}(P) \simeq \operatorname{Hom}_{\mathcal{O}_X}(P, M)$ for all $P \in \mathsf{D}^b_{\mathsf{Coh}}(X)$. Standard properties of adjoint functors show that the assignment $\mathcal{M} \mapsto M$ defines a right adjoint $c_*\colon \mathsf{D}^b_{\mathsf{Coh}}(\mathcal{X}) \to \mathsf{D}^b_{\mathsf{Coh}}(X)$ to $c^*\colon \mathsf{D}^b_{\mathsf{Coh}}(X) \to \mathsf{D}^b_{\mathsf{Coh}}(\mathcal{X})$.

Our next observation is the following: if $P \in \mathsf{D}^b_{\mathsf{Coh}}(X)$ and $\mathcal{M} \in \mathsf{D}^b_{\mathsf{Coh}}(\mathcal{X})$, then there is always a projection formula:

$$(c_*\mathcal{M}) \otimes^{\mathsf{L}}_{\mathcal{O}_X} P \simeq c_*(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} c^*P).$$

This follows from the smoothness of X (so every object of $\mathsf{D}^b_{\mathsf{Coh}}(X)$ is perfect and dualizable) and abstract nonsense about adjoints and dualizables (see Lemma 4.3).

For the full faithfulness of c^*, it is sufficient to prove that $\eta_N\colon N \to c_*c^*N$ is a quasi-isomorphism for all $N \in \mathsf{D}^b_{\mathsf{Coh}}(X)$. Let H_N be a cone of η_N. By Nakayama's Lemma, it is sufficient to prove that $H_N \otimes^{\mathsf{L}}_{\mathcal{O}_X} \kappa(y) \simeq 0$ for all closed points $y \in X$.

Fix a closed point $y \in X$. The projection formula shows that $H_N \otimes^{\mathsf{L}}_{\mathcal{O}_X} \kappa(y) \simeq H_{N\otimes^{\mathsf{L}}_{\mathcal{O}_X}\kappa(y)}$. But $N \otimes^{\mathsf{L}}_{\mathcal{O}_X} \kappa(y)$ is just a direct sum of shifts of $\kappa(y)$. Hence, we are reduced to proving that $\eta_{\kappa(y)}$ is a quasi-isomorphism. An elementary argument using stalks shows that $c^*\kappa(y) \simeq \kappa(x)$ in $\mathsf{D}^b_{\mathsf{Coh}}(\mathcal{X})$ for some unique closed point $x \in \mathcal{X}$ (see the proof of Theorem A). Thus, it remains to prove that $\kappa(y) \to c_*\kappa(x)$ is a quasi-isomorphism. If $y' \neq y$ is a closed point, then again there is a unique closed point $x' \in \mathcal{X}$ such that $c^*\kappa(y') \simeq \kappa(x')$ in $\mathsf{D}^b_{\mathsf{Coh}}(\mathcal{X})$. The projection formula shows us that:

$$(c_*\kappa(x)) \otimes^{\mathsf{L}}_{\mathcal{O}_X} \kappa(y') \simeq c_*(\kappa(x) \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \kappa(x')) \simeq 0.$$

It follows that $c_*\kappa(x)$ is supported only at y. In particular, we can determine $c_*\kappa(x)$ from its global sections. But

$$\mathsf{R}\Gamma(X, c_*\kappa(x)) = \mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, c_*\kappa(x)) \simeq \mathsf{R}\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}, \kappa(x)) \simeq \mathsf{R}\Gamma(\mathcal{X}, \kappa(x)) \simeq \kappa(x).$$

Since $\kappa(y) \to \kappa(x)$ is an isomorphism, we conclude that $\eta_{\kappa(y)}$ is a quasi-isomorphism and c^* is fully faithful.

For the essential surjectivity of $c^* \colon \mathsf{D}^b_{\mathsf{Coh}}(X) \to \mathsf{D}^b_{\mathsf{Coh}}(\mathcal{X})$, let $\mathcal{M} \in \mathsf{D}^b_{\mathsf{Coh}}(\mathcal{X})$. Set $M = c_*\mathcal{M} \in \mathsf{D}^b_{\mathsf{Coh}}(X)$. We will prove that M algebraizes $\mathcal{M}$; that is, the adjunction $\epsilon_{\mathcal{M}} \colon c^*c_*\mathcal{M} \to \mathcal{M}$ is a quasi-isomorphism. Let $E_{\mathcal{M}}$ be a cone for $\epsilon_{\mathcal{M}}$. Again by Nakayama's Lemma, it is sufficient to prove that $E_{\mathcal{M}} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \kappa(x) \simeq 0$ for all closed points $x \in \mathcal{X}$. Now we have already seen that if $x \in \mathcal{X}$ is closed, then $\kappa(x) \simeq c^*\kappa(x')$ for a unique closed point x' of X. In particular, it follows that it is sufficient to prove that $E_{\mathcal{M}} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} c^*\kappa(x') \simeq 0$ for all closed points x' of X. But the projection formula shows that we have quasi-isomorphisms:

$$(c^*c_*\mathcal{M}) \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} c^*\kappa(x') \simeq c^*(c_*\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \kappa(x')) \simeq c^*c_*(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} c^*\kappa(x')).$$

It follows immediately that $E_{\mathcal{M}} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} c^*\kappa(x') \simeq E_{\mathcal{M}\otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} c^*\kappa(x')}$. Again, $\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} c^*\kappa(x')$ is just a finite direct sum of shifts of $c^*\kappa(x')$. Hence, it remains to prove that $c^*c_*c^*\kappa(x') \to c^*\kappa(x')$ is a quasi-isomorphism. But we've already seen that $\kappa(x') \to c_*c^*\kappa(x')$ is a quasi-isomorphism, and the result follows.

Note that the full faithfulness of $c^* \colon \mathsf{D}^b_{\mathsf{Coh}}(X) \to \mathsf{D}^b_{\mathsf{Coh}}(\mathcal{X})$ implies the cohomological comparison result. Indeed, if $F \in \mathsf{Coh}(X)$, then

$$\begin{aligned}\mathrm{H}^i(X, F) &\simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, F[i]) \simeq \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(c^*\mathcal{O}_X, c^*F[i]) \\ &\simeq \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}, c^*F[i]) \simeq \mathrm{H}^i(\mathcal{X}, c^*F).\end{aligned}$$

2.2. **Lefschetz.** We will prove the following variant of Theorem B, which is sufficient to establish the full-faithfulness of the analytification of vector bundles on the Fargues–Fontaine curve, which is a regular noetherian scheme of dimension 1 that is universally closed and separated but not locally of finite type over $\operatorname{Spec} \mathbb{Q}_p$.

Theorem 2.1. *Let X be a regular noetherian scheme of dimension* 1. *Let $c \colon \mathcal{X} \to X$ be a morphism of locally ringed spaces. Let $x \in X$ be a closed point. Assume*

(1) *$U = X - \{x\}$ is quasi-affine;*
(2) *$c^{-1}(x)$ consists of a single point y;*
(3) *$\mathcal{O}_{X,x} \to \mathcal{O}_{\mathcal{X},y}$ is flat and $\kappa(x) \to \kappa(x) \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{\mathcal{X},y}$ is an isomorphism; and*
(4) *$\mathrm{H}^0(X, \mathcal{O}_X) \simeq \mathrm{H}^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $\mathrm{H}^1(X, \mathcal{O}_X) \hookrightarrow \mathrm{H}^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.*

Then the comparison morphism:

$$\mathrm{H}^0(X, F) \to \mathrm{H}^0(\mathcal{X}, c^*F)$$

is an isomorphism for all vector bundles F on X.

Proof. With no finiteness at our disposal, we have to work with unbounded derived categories of sheaves. Specifically, we consider the adjoint pair:

$$\mathsf{L}c^* \colon \mathsf{D}_{\mathrm{qc}}(X) \leftrightarrows \mathsf{D}(\mathcal{X}) \colon \mathsf{R}c_{\mathrm{qc},*}.$$

These will be discussed in great detail in §4. If $N \in \mathsf{D}_{\mathrm{qc}}(X)$, then we have the adjunction $\eta_N \colon N \to \mathsf{R}c_{\mathrm{qc},*}\mathsf{L}c^*N$; let Q_N denote a cone for this morphism. We must prove that if F is a vector bundle on X, then $\tau^{\leq 0}Q_F \simeq 0$. Observe that the projection formula implies that for a coherent sheaf N on X we have $Q_{\mathcal{O}_X} \otimes^{\mathsf{L}}_{\mathcal{O}_X} N \simeq Q_N$ (this is because X is regular, so N is perfect). Set $Q = Q_{\mathcal{O}_X}$; hence, it suffices to prove that $\tau^{\leq 0}Q_{\mathcal{O}_X} \simeq 0$. Conditions (2) and (3), just like in the previous section, combine to show that $Q \otimes^{\mathsf{L}}_{\mathcal{O}_X} \kappa(x) \simeq 0$. Now let $j \colon U \hookrightarrow X$ be the resulting open immersion; then localization theory (e.g., [HR17b, Ex. 1.4]) now implies that $Q \simeq \mathsf{R}j_*\mathsf{L}j^*Q$. But U is quasi-affine, so it follows that $\tau^{\leq 0}Q \simeq 0$ if and only if $\tau^{\leq 0}\mathsf{R}\Gamma(X, Q) \simeq 0$. We now have the long exact sequence:

$$0 \to \mathrm{H}^0(X, Q) \to \mathrm{H}^0(X, \mathcal{O}_X) \to \mathrm{H}^0(\mathcal{X}, \mathcal{O}_X) \to \mathrm{H}^0(X, Q) \to \mathrm{H}^1(X, \mathcal{O}_X) \to \mathrm{H}^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}).$$

Certainly, $\tau^{<-1}Q \simeq 0$, so we have $\mathrm{H}^{-1}(X, Q) \simeq \mathrm{H}^0(X, \mathcal{H}^{-1}(Q))$. But $\mathrm{H}^0(X, \mathcal{O}_X) \simeq \mathrm{H}^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, so $\mathrm{H}^0(X, \mathcal{H}^{-1}(Q)) = 0$ and $\mathcal{H}^{-1}(Q) \simeq 0$. Hence, $\mathrm{H}^0(X, Q) \simeq \mathrm{H}^0(X, \mathcal{H}^0(Q))$. The injection $\mathrm{H}^1(X, \mathcal{O}_X) \to \mathrm{H}^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ allows us to conclude that $\mathcal{H}^0(Q) \simeq 0$ too. The result follows. □

Remark 2.2. Assume that we are in the situation of Theorem 2.1. Let X_{cl} be the set of closed points of X and let $\mathcal{X}_{\mathrm{cl},c} = c^{-1}(X_{\mathrm{cl}})$. Additionally, assume the following:

(5) c_* preserves direct limits of $\mathcal{O}_{\mathcal{X}}$-modules; and
(6) $\mathcal{X}$ has the following property: given a direct system of vector bundles with injective transition maps $\{\mathcal{E}_n\}_{n\geq 0}$ such that $\varinjlim_n \mathcal{E}_n \otimes_{\mathcal{O}_{\mathcal{X}}} \kappa(y) \simeq 0$ and $\varinjlim_n \Gamma(\mathcal{X}, \mathcal{E}_n) \simeq 0$, then $\mathcal{E}_n = 0$ for all n.

Condition (6) holds if X is a scheme (by support theory), but in the adic/analytic setting seems difficult (or false). We sketch an argument that the conditions above imply that $c_{\mathrm{qc},*}$ is conservative on vector bundles; this is not quite enough to deduce that $c^* \colon \mathsf{Vect}(X) \to \mathsf{Vect}(\mathcal{X})$ is an equivalence, but it is very close. If $\mathcal{V} \in \mathsf{Vect}(\mathcal{X})$, consider $\mathcal{V} \otimes_{\mathcal{O}_{\mathcal{X}}} c^*j_*\mathcal{O}_U$. Observe that $j_*\mathcal{O}_U = \varinjlim_n L^n$, where $L = \ker(\mathcal{O}_X \to \kappa(x))$. By the projection formula, $c_{\mathrm{qc},*}(\mathcal{V} \otimes_{\mathcal{O}_{\mathcal{X}}} c^*j_*\mathcal{O}_U) \simeq c_{\mathrm{qc},*}(\mathcal{V}) \otimes_{\mathcal{O}_X} j_*\mathcal{O}_U$. Now take $\mathcal{E}_n = \mathcal{V} \otimes_{\mathcal{O}_{\mathcal{X}}} c^*L^n$; then

$$\varinjlim_n \mathcal{E}_n \otimes \kappa(y) \simeq \mathcal{V} \otimes_{\mathcal{O}_{\mathcal{X}}} c^*(j_*\mathcal{O}_U \otimes_{\mathcal{O}_X} \kappa(x)) \simeq 0.$$

Now if $c_{\mathrm{qc},*}(\mathcal{V}) \simeq 0$, then it follows immediately from the projection formula calculation above that $\varinjlim_n \Gamma(\mathcal{X}, \mathcal{E}_n) = 0$ too. By (6), $\mathcal{E}_n = 0$ for all n; hence, $\mathcal{V} = 0$.

3. A finiteness result

Our first task is to consider a variant of the finiteness result [BB03, Thm. 1.1] for non-noetherian algebraic spaces. This was recently established in the noetherian case in [BZNP17] and in general in [Stacks], where it is formulated in terms of pseudo-coherence [SGA6]. Since the non-noetherian situation will be important to us, we will briefly recall these ideas.

Let B be a ring. A bounded complex of finitely generated and projective B-modules is called *strictly perfect*. Let $m \in \mathbb{Z}$. A complex of B-modules M is

m-pseudo-coherent if there is a morphism $\phi\colon P \to M$ such that P is strictly perfect and the induced morphism $\mathcal{H}^i(\phi)\colon \mathcal{H}^i(P) \to \mathcal{H}^i(P)$ is an isomorphism for $i > m$ and surjective for $i = m$. A complex of B-modules M is *pseudo-coherent* if it is m-pseudo-coherent for all integers $m \in \mathbb{Z}$; equivalently, it is quasi-isomorphic to a bounded above complex of finitely generated and projective B-modules [Stacks, Tag 064T]. These conditions are all stable under derived base change [Stacks, Tag 0650] and are flat local [Stacks, Tag 068R].

We let $\mathsf{D}^-_{\mathrm{pc}}(B)$ denote the full triangulated subcategory of the derived category of B-modules, $\mathsf{D}(B)$, with objects those complexes of B-modules that are quasi-isomorphic to a pseudo-coherent complex of B-modules. We let $\mathsf{D}^b_{\mathrm{pc}}(B) \subseteq \mathsf{D}^-_{\mathrm{pc}}(B)$ be the triangulated subcategory of objects with bounded cohomological support. If $B \to C$ is a ring homomorphism, then derived base change induces $- \otimes^{\mathsf{L}}_B C\colon \mathsf{D}^-_{\mathrm{pc}}(B) \to \mathsf{D}^-_{\mathrm{pc}}(C)$. If C has finite tor-dimension over B (e.g., C is B-flat), then the derived base change sends bounded pseudo-coherent complexes to bounded pseudo-coherent complexes.

The above generalizes to ringed sites [Stacks, Tag 08FS]. Let $\mathcal{X}$ be a ringed site. A complex of $\mathcal{O}_{\mathcal{X}}$-modules is *strictly perfect* if it is bounded and each term is a direct summand of a finitely generated and free $\mathcal{O}_{\mathcal{X}}$-module [Stacks, Tag 08FL].

Example 3.1. Let X be an algebraic space. If $i\colon D \subseteq X$ is a Cartier divisor, then $i_*\mathcal{O}_D \in \mathsf{D}_{\mathrm{qc}}(X)$ is perfect. More generally, if $i\colon Z \hookrightarrow X$ is a regular embedding (i.e., i is locally the zero locus of a regular section of a vector bundle), then $i_*\mathcal{O}_Z \in \mathsf{D}_{\mathrm{qc}}(X)$ is perfect. Also, if X is quasi-compact and quasi-separated and $j\colon U \subseteq X$ is a quasi-compact open immersion, then there is a perfect complex $P \in \mathsf{D}_{\mathrm{qc}}(X)$ whose cohomological support is precisely $X \setminus U$ [HR17a, Thm. A].

Let $m \in \mathbb{Z}$. A complex of $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{M}$ is *m-pseudo-coherent* if locally on $\mathcal{X}$ there is a morphism $\phi\colon \mathcal{P} \to \mathcal{M}$ such that $\mathcal{P}$ is strictly perfect and the induced morphism $\mathcal{H}^i(\phi)\colon \mathcal{H}^i(\mathcal{P}) \to \mathcal{H}^i(\mathcal{M})$ is an isomorphism for $i > m$ and surjective for $i = m$. A complex of $\mathcal{O}_{\mathcal{X}}$-modules is *pseudo-coherent* if it is m-pseudo-coherent for every $m \in \mathbb{Z}$.

Let $\mathsf{D}^-_{\mathrm{pc}}(\mathcal{X})$ denote the full triangulated subcategory of $\mathsf{D}(\mathcal{X})$, the unbounded derived category of $\mathcal{O}_{\mathcal{X}}$-modules, with objects those complexes that are quasi-isomorphic to a bounded above pseudo-coherent complex. We let $\mathsf{D}^b_{\mathrm{pc}}(\mathcal{X}) \subseteq \mathsf{D}^-_{\mathrm{pc}}(\mathcal{X})$ be the full triangulated subcategory of objects with bounded cohomological support. If $c\colon \mathcal{X} \to X$ is a morphism of ringed sites, then the restriction of $\mathsf{L}c^*\colon \mathsf{D}(X) \to \mathsf{D}(\mathcal{X})$ to $\mathsf{D}^-_{\mathrm{pc}}(X)$ factors through $\mathsf{D}^-_{\mathrm{pc}}(\mathcal{X})$ [Stacks, Tag 08H4]. Moreover, if c has finite tor-dimension (e.g., it is flat), then $\mathsf{L}c^*$ preserves bounded complexes.

Example 3.2. Perfect complexes on sites (i.e., complexes that are locally strictly perfect) are pseudo-coherent. In particular, vector bundles of finite rank are pseudo-coherent.

Example 3.3. Let $\mathcal{X}$ be a ringed site with a coherent structure sheaf. For example, a locally noetherian algebraic space or an analytic space. Let $* \in \{-, b\}$. Then $\mathsf{D}^*_{\mathrm{pc}}(\mathcal{X}) = \mathsf{D}^*_{\mathsf{Coh}}(\mathcal{X})$; that is, a complex $M \in \mathsf{D}(\mathcal{X})$ is pseudo-coherent if and only if it is quasi-isomorphic to a bounded above complex of sheaves with coherent cohomology [SGA6, Cor. I.3.5].

The following lemma improves upon those given in Example 3.1 in the coherent situation. To state this lemma, we recall the following definition [Nee01, §2.1].

Let $\mathcal{T}$ be a triangulated category. A subcategory $\mathcal{S} \subseteq \mathcal{T}$ is *thick* (or épaisse) if it is triangulated and is closed under direct summands. If $S \subseteq \mathcal{T}$ is a collection of objects, we let $\langle S \rangle \subseteq \mathcal{T}$ denote the thick closure of S; that is, it is the smallest thick triangulated subcategory of $\mathcal{T}$ containing S.

Lemma 3.4. *Let X be a quasi-compact and quasi-separated algebraic space. Let $i\colon Z \hookrightarrow X$ be a finitely presented closed immersion. Let*

$$\mathsf{D}^b_{\mathrm{pc},|Z|}(X) = \ker(\mathsf{L}j^*\colon \mathsf{D}^b_{\mathrm{pc}}(X) \to \mathsf{D}^b_{\mathrm{pc}}(U)),$$

where $j\colon U \to X$ is the open immersion complementary to $i\colon Z \to X$. If $\mathcal{O}_X$ is coherent, then $\langle \mathsf{R}i_*\mathsf{D}^b_{\mathrm{pc}}(Z)\rangle = \mathsf{D}^b_{\mathrm{pc},|Z|}(X)$.

Proof. Clearly, $\mathsf{R}i_*\mathsf{D}^b_{\mathrm{pc}}(Z) \subseteq \mathsf{D}^b_{\mathrm{pc},|Z|}(X)$. Since $\mathsf{D}^b_{\mathrm{pc},|Z|}(X)$ is a thick subcategory of $\mathsf{D}^b_{\mathrm{pc}}(X)$, it follows that $\langle \mathsf{R}i_*\mathsf{D}^b_{\mathrm{pc}}(Z)\rangle \subseteq \mathsf{D}^b_{\mathrm{pc},|Z|}(X)$. For the reverse inclusion, by induction on the length of a complex, it is sufficient to prove that if $M \in \mathsf{Coh}_{|Z|}(X) = \ker(j^*\colon \mathsf{Coh}(X) \to \mathsf{Coh}(U))$ then $M \in \langle \mathsf{R}i_*\mathsf{D}^b_{\mathrm{pc}}(Z)\rangle$. Let $I = \ker(\mathcal{O}_X \to i_*\mathcal{O}_Z)$. By [HR16, Lem. 2.5(i)], it follows that there exists an integer $n > 0$ such that $I^{n+1}M = 0$. Hence, M admits a finite filtration by $i_*\mathcal{O}_Z$-modules and so belongs to $\langle \mathsf{R}i_*\mathsf{D}^b_{\mathrm{pc}}(Z)\rangle$. $\square$

Let A be a ring. A B-algebra A is *pseudo-coherent* if it admits a surjection from a polynomial ring $\phi\colon A[x_1,\dots,x_n] \twoheadrightarrow B$ such that B is a pseudo-coherent $A[x_1,\dots,x_n]$-module. Pseudo-coherence is stable under flat base change on A and is étale local on B. See [Stacks, Tag 067X] for more background material. This definition generalizes readily to morphisms of algebraic spaces [Stacks, Tag 06BQ]. We now recall some examples that will be important to us.

Example 3.5. Let A be a noetherian ring. If $X \to \operatorname{Spec} A$ is a locally of finite type morphism of algebraic spaces, then it is pseudo-coherent [Stacks, Tag 06BX].

Example 3.6. Let A be a ring. If $X \to \operatorname{Spec} A$ is a flat and locally of finite presentation morphism of algebraic spaces, then it is pseudo-coherent [Stacks, Tag 06BV].

Example 3.7. Let A be a universally cohesive ring. That is, every finitely presented A-algebra is a coherent ring. The standard example is an a-adically complete valuation ring; for example, $A = \mathcal{O}_{\mathbb{C}_p}$, the ring of integers in the p-adically completed algebraic closure of $\mathbb{Q}_p$, $\mathbb{C}_p$. If $X \to \operatorname{Spec} A$ is a locally of finite presentation morphism of algebraic spaces, then it is pseudo-coherent. This is the setting for Fujiwara–Kato's formalism of rigid geometry [FK13].

The main result of this section is the following small refinement of [Stacks, Tag 0CTT].

Theorem 3.8. *Let A be a ring. Let $X \to \operatorname{Spec} A$ be a quasi-compact, separated, and pseudo-coherent morphism of algebraic spaces. Let $M \in \mathsf{D}_{\mathrm{qc}}(X)$. If $\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(P, M) \in \mathsf{D}(A)$ is pseudo-coherent (pseudo-coherent and bounded) for all perfect complexes P, then M is pseudo-coherent (pseudo-coherent and bounded).*

Proof. It is sufficient to prove that the condition implies that $\mathsf{R}\Gamma(X, E \otimes^{\mathsf{L}}_{\mathcal{O}_X} M) \in \mathsf{D}(A)$ is pseudo-coherent for all pseudo-coherent E on X. Indeed, pseudo-coherent morphisms are locally finitely presented, so M is pseudo-coherent relative to A

[Stacks, Tag 0CTT]. Since $X \to \operatorname{Spec} A$ is pseudo-coherent, M is pseudo-coherent on X [Stacks, Tag 0DHQ]. The boundedness result is [BZNP17, Lem. 3.0.14].

We begin by observing that if $P \in \mathsf{D}_{\mathrm{qc}}(X)$ is perfect, then $\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(P^\vee, M) \simeq \mathsf{R}\Gamma(X, P \otimes^{\mathsf{L}}_{\mathcal{O}_X} M) \in \mathsf{D}(A)$, which is pseudo-coherent by assumption. Now there exists an integer n such that $\mathrm{H}^r(X, N) = 0$ for all $r > n$ and $N \in \mathsf{QCoh}(X)$ [Stacks, Tag 073G]. By [LO08, Rem. 2.1.11], $\tau^{\geq m}\mathsf{R}\Gamma(X, G) \simeq \tau^{\geq m}\mathsf{R}\Gamma(X, \tau^{\geq l}G)$ for all $m \in \mathbb{Z}$, $l \leq m - n$, and $G \in \mathsf{D}_{\mathrm{qc}}(X)$. Let $E \in \mathsf{D}_{\mathrm{qc}}(X)$ be pseudo-coherent and fix $m \in \mathbb{Z}$. Let $j \in \mathbb{Z}$ be such that $\tau^{>j}M \simeq 0$ and $\tau^{>j}E \simeq 0$. Choose a perfect complex $P \in \mathsf{D}_{\mathrm{qc}}(X)$ and morphism $\phi\colon P \to E$ such that $\tau^{\geq a}\mathrm{cone}(\phi) \simeq 0$, where $a = m - n - j$ [Stacks, Tag 08HP]. Then

$$\begin{aligned}
\tau^{\geq m}\mathsf{R}\Gamma(X, E \otimes^{\mathsf{L}}_{\mathcal{O}_X} M) &\simeq \tau^{\geq m}\mathsf{R}\Gamma(X, \tau^{\geq m-n}(E \otimes^{\mathsf{L}}_{\mathcal{O}_X} M)) \\
&\simeq \tau^{\geq m}\mathsf{R}\Gamma(X, \tau^{\geq m-n}((\tau^{\geq m-n-j}E) \otimes^{\mathsf{L}}_{\mathcal{O}_X} M)) \\
&\simeq \tau^{\geq m}\mathsf{R}\Gamma(X, \tau^{\geq m-n}((\tau^{\geq m-n-j}P) \otimes^{\mathsf{L}}_{\mathcal{O}_X} M) \\
&\simeq \tau^{\geq m}\mathsf{R}\Gamma(X, \tau^{\geq m-n}(P \otimes^{\mathsf{L}}_{\mathcal{O}_X} M)) \\
&\simeq \tau^{\geq m}\mathsf{R}\Gamma(X, P \otimes^{\mathsf{L}}_{\mathcal{O}_X} M).
\end{aligned}$$

We have already seen that $\mathsf{R}\Gamma(X, P \otimes^{\mathsf{L}}_{\mathcal{O}_X} M) \in \mathsf{D}(A)$ is pseudo-coherent, and the claim follows. □

Remark 3.9. Theorem 3.8 has a converse if $X \to \operatorname{Spec} A$ is proper. If M is pseudo-coherent (resp. pseudo-coherent and bounded) and P is perfect, then $\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(P, M) \simeq \mathsf{R}\Gamma(X, P^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_X} M)$. Replacing M by $P^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_X} M$, it suffices to prove that $\mathsf{R}\Gamma(X, -)$ sends pseudo-coherent complexes to pseudo-coherent complexes. If A is noetherian, this is just the usual coherence theorem for algebraic spaces [Knu71, Thm. IV.4.1]. If X is a scheme and A is not necessarily noetherian, this is Kiehl's Finiteness Theorem [Kie72, Thm. 2.9]. If $X \to \operatorname{Spec} A$ is flat, this is in the Stacks Project [Stacks, Tag 0CSC]. If A is universally cohesive, then this is due to Fujiwara–Kato [FK13, Thm. 8.1.2]. Using derived algebraic geometry, the argument given in the Stacks Project readily extends to the general (i.e., non-flat) situation; that is, a version of Kiehl's finiteness theorem for algebraic spaces. This is done by Lurie in [SAG].

As noted in [BZNP17, Rem. 3.0.6], it is Theorem 3.8 that fails miserably for algebraic stacks with infinite stabilizers. In future work, we will describe a variant of Theorem 3.8 for a large class of algebraic stacks with infinite stabilizers that is sufficient to establish integral transform and GAGA results.

We conclude this section with a simple corollary of Theorem 3.8. Variants of this are well-known (see [Lip09, Ex. 4.3.9] and [Ryd14] in the finite type noetherian, but non-separated situation).

Corollary 3.10. *Let A be a universally cohesive ring (e.g., noetherian). Let $X \to \operatorname{Spec} A$ be a quasi-compact and separated morphism of algebraic spaces. If $\mathsf{R}\Gamma(X, -)$ sends $\mathsf{D}^b_{\mathrm{pc}}(X)$ to $\mathsf{D}^b_{\mathrm{pc}}(A)$, then $X \to \operatorname{Spec} A$ is proper and of finite presentation.*

Proof. By absolute noetherian approximation [Ryd15], there is an affine morphism $a\colon X \to X_0$, where X_0 is a separated and finitely presented algebraic space over $\operatorname{Spec} A$. Using Nagata's compactification theorem for algebraic spaces [CLO12], a blow-up, and absolute noetherian approximation again, we may further assume that $X_0 \to \operatorname{Spec} A$ is proper and finitely presented. Since A is universally cohesive,

$\mathcal{O}_{X_0}$ is coherent. Now let $P \in \mathsf{D}_{\mathrm{qc}}(X_0)$ be a perfect complex; then

$$\mathsf{R}\mathrm{Hom}_{\mathcal{O}_{X_0}}(P, a_*\mathcal{O}_X) = \mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathsf{L}a^*P, \mathcal{O}_X) = \mathsf{R}\Gamma(X, \mathsf{L}a^*P^\vee) \in \mathsf{D}^b_{\mathsf{Coh}}(A).$$

Hence, Theorem 3.8 implies that $a_*\mathcal{O}_X \in \mathsf{Coh}(X_0)$. That is, a is finite and finitely presented. By composition, $X \to \operatorname{Spec} A$ is proper and of finite presentation. □

4. Adjoints

Throughout we let X be an algebraic space. Consider a morphism of ringed topoi $c\colon \mathcal{X} \to X_{\text{ét}}$. There is an adjoint pair on the level of unbounded derived categories

$$\mathsf{L}c^*\colon \mathsf{D}(X) \rightleftarrows \mathsf{D}(\mathcal{X})\colon \mathsf{R}c_*.$$

The inclusion $\mathsf{D}_{\mathrm{qc}}(X) \subseteq \mathsf{D}(X)$ is fully faithful and also admits a right adjoint, the quasi-coherator $\mathsf{R}Q_X\colon \mathsf{D}(X) \to \mathsf{D}_{\mathrm{qc}}(X)$. It follows immediately that

(1) the restriction of $\mathsf{L}c^*$ to $\mathsf{D}_{\mathrm{qc}}(X)$ is left adjoint to $\mathsf{R}Q_X\mathsf{R}c_*$; and
(2) if $M \in \mathsf{D}_{\mathrm{qc}}(X)$, then the natural map $M \to \mathsf{R}Q_X(M)$ is a quasi-isomorphism.

We will let

$$\mathsf{L}c^*\colon \mathsf{D}_{\mathrm{qc}}(X) \rightleftarrows \mathsf{D}(\mathcal{X})\colon \mathsf{R}c_{\mathrm{qc},*}$$

denote the resulting adjoint pair. Let $M \in \mathsf{D}_{\mathrm{qc}}(X)$ and $\mathcal{N} \in \mathsf{D}(\mathcal{X})$. Let

$$\eta_M\colon M \to \mathsf{R}c_{\mathrm{qc},*}\mathsf{L}c^*M \quad \text{and} \quad \epsilon_{\mathcal{N}}\colon \mathsf{L}c^*\mathsf{R}c_{\mathrm{qc},*}\mathcal{N} \to \mathcal{N}$$

denote the morphisms resulting from the adjunctions.

We now use Theorem 3.8 to show that $\mathsf{R}c_{\mathrm{qc},*}$ frequently preserves pseudo-coherence.

Proposition 4.1. *Let A be a ring. Let $X \to \operatorname{Spec} A$ be a quasi-compact, separated, and pseudo-coherent morphism of algebraic spaces. Let $c\colon \mathcal{X} \to X_{\text{ét}}$ be a morphism of ringed topoi. Let $* \in \{b, -\}$. If $\mathsf{R}\Gamma(\mathcal{X}, -)$ sends $\mathsf{D}^*_{\mathrm{pc}}(\mathcal{X})$ to $\mathsf{D}^*_{\mathrm{pc}}(A)$, then the restriction of $\mathsf{R}c_{\mathrm{qc},*}$ to $\mathsf{D}^*_{\mathrm{pc}}(\mathcal{X})$ factors through $\mathsf{D}^*_{\mathrm{pc}}(X)$.*

Proof. Let $\mathcal{M} \in \mathsf{D}^*_{\mathrm{pc}}(\mathcal{X})$ and let $P \in \mathsf{D}_{\mathrm{qc}}(X)$ be perfect. Then $\mathsf{L}c^*P^\vee \in \mathsf{D}_{\mathrm{qc}}(\mathcal{X})$ is perfect, so $\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathsf{L}c^*P \in \mathsf{D}^*_{\mathrm{pc}}(\mathcal{X})$. Hence,

$$\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(P, \mathsf{R}c_{\mathrm{qc},*}\mathcal{M}) \simeq \mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathsf{L}c^*P, \mathcal{M}) \simeq \mathsf{R}\Gamma(\mathcal{X}, \mathsf{L}c^*P^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{M}) \in \mathsf{D}^*_{\mathrm{pc}}(A).$$

By Theorem 3.8, $\mathsf{R}c_{\mathrm{qc},*}\mathcal{M} \in \mathsf{D}^*_{\mathrm{pc}}(X)$. □

Remark 4.2. A variant of Proposition 4.1 that is valid for finite type cohomological functors for proper schemes over noetherian bases, which generalizes [BB03, Thm. 1.1], appears in [Nee18].

We now return to our general discussion. The categories $\mathsf{D}_{\mathrm{qc}}(X)$ and $\mathsf{D}(\mathcal{X})$ are symmetric monoidal and the derived pullback $\mathsf{L}c^*$ is strong monoidal. This lets us apply the formalism in Appendix A to our situation. We record some consequences here.

Lemma 4.3. *If $M \in \mathsf{D}_{\mathrm{qc}}(X)$ and $\mathcal{N} \in \mathsf{D}(\mathcal{X})$, then there is a natural projection morphism*

$$\pi_{M,\mathcal{N}}\colon M \otimes^{\mathsf{L}}_{\mathcal{O}_X} (\mathsf{R}c_{\mathrm{qc},*}\mathcal{N}) \to \mathsf{R}c_{\mathrm{qc},*}(\mathsf{L}c^*M \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{N}).$$

This is an isomorphism if M is perfect or $\mathcal{O}_{\mathcal{X}}$ is a compact object of $\mathsf{D}(\mathcal{X})$.

Proof. If M is perfect, then it is a dualizable object of $\mathsf{D}_{\mathrm{qc}}(X)$ [HR17a, Lem. 4.3]. Hence, the projection morphism is an isomorphism in this case by Theorem A.8. If $\mathcal{O}_{\mathcal{X}}$ is a compact object of $\mathsf{D}(\mathcal{X})$, then $\mathsf{L}c^*\colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}(\mathcal{X})$ preserves compact objects [HR17b, Lem. 2.3(2)], so its right adjoint $\mathsf{R}c_{\mathrm{qc},*}$ preserves small coproducts [Nee96, Thm. 5.1]. Hence, the full subcategory of $\mathsf{D}_{\mathrm{qc}}(X)$ consisting of those M for which $\pi_{M,\mathcal{N}}$ is an isomorphism is localizing and contains the perfect complexes. By Thomason's localization theorem [Nee96, Thm. 2.1.2], the result follows. □

Remark 4.4. The condition that $\mathcal{O}_{\mathcal{X}}$ is a compact object of $\mathsf{D}(\mathcal{X})$ is subtle, but frequently satisfied. A useful criterion is [Stacks, Tag 094D], which shows that it is sufficient for cohomology of abelian sheaves on $\mathcal{X}$ to commute with filtered colimits and have finite cohomological dimension. It follows that $\mathcal{O}_{\mathcal{X}}$ is a compact object of $\mathsf{D}(\mathcal{X})$ whenever

(1) $\mathcal{X}$ is equivalent to the topos of a noetherian topological space of finite Krull dimension (Grothendieck's Theorem, see [Gro57, Thm. 3.6.5] and [Stacks, Tag 02UZ]);
(2) $\mathcal{X}$ is equivalent to the topos of a spectral topological space of finite Krull dimension (this generalizes the above, see [Sch92, Thm. 4.5] and [Stacks, Tag 0A3G];
(3) $\mathcal{X}$ is equivalent to the topos of a compact Hausdorff space of finite cohomological dimension (e.g., it has finite covering dimension).

5. Equivalences

In this section, the following setup will feature frequently.

Setup 5.1. Let X be an algebraic space. Let $c\colon \mathcal{X} \to X_{\mathrm{\acute{e}t}}$ be a morphism of ringed topoi. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. Let Z and $\mathcal{Z}$ be the ringed topoi $(X_{\mathrm{\acute{e}t}}, \mathcal{A})$ and $(\mathcal{X}, c^*\mathcal{A})$, respectively. There is an induced 2-commutative diagram of ringed topoi:

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "c_Z"] \arrow[d, "i'"'] & Z \arrow[d, "i"] \\ \mathcal{X} \arrow[r, "c"] & X_{\mathrm{\acute{e}t}}. \end{tikzcd}$$

Assume that

$$\mathsf{L}c_Z^*\colon \mathsf{D}^-_{\mathrm{pc}}(Z) \to \mathsf{D}^-_{\mathrm{pc}}(\mathcal{Z})$$

is an equivalence.

Remark 5.2. In Setup 5.1, if $\tilde{Z}$ denotes the algebraic space defined by the quasi-coherent sheaf of algebras $\mathcal{A}$, then there is a natural equivalence $\mathsf{D}_{\mathrm{qc}}(Z) \simeq \mathsf{D}_{\mathrm{qc}}(\tilde{Z})$ [HR17a, Cor. 2.7]. This equivalence restricts to one on pseudo-coherent complexes.

Remark 5.3. In practice, the equivalence $\mathsf{L}c_Z^*$ in Setup 5.1 is often quite easy to check. For example, when the underlying topoi of $\mathcal{X}$ and $X_{\mathrm{\acute{e}t}}$ are equivalent and $\mathcal{A} \to \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathcal{X}}$ is an isomorphism of $\mathcal{O}_X$-algebras. This always the case when c is a formal completion along a closed immersion.

The following two lemmas provide a sanity check in Setup 5.1.

Lemma 5.4. *Assume Setup 5.1. The restriction of* $\mathsf{R}c_{Z,\mathrm{qc},*}$ *to* $\mathsf{D}^-_{\mathrm{pc}}(\mathcal{Z})$ *induces an equivalence:*

$$\mathsf{L}c_Z^*\colon \mathsf{D}^-_{\mathrm{pc}}(Z) \leftrightarrows \mathsf{D}^-_{\mathrm{pc}}(\mathcal{Z})\colon \mathsf{R}c_{Z,\mathrm{qc},*}.$$

Proof. Let P, $Q \in \mathsf{D}^-_{\mathrm{pc}}(Z)$. Then there are isomorphisms:

$$\mathsf{RHom}_{\mathcal{O}_Z}(P,Q) \simeq \mathsf{RHom}_{\mathcal{O}_{\mathcal{Z}}}(\mathsf{L}c_Z^*P, \mathsf{L}c_Z^*Q) \simeq \mathsf{RHom}_{\mathcal{O}_Z}(P, \mathsf{R}c_{Z,\mathrm{qc},*}\mathsf{L}c_Z^*Q).$$

Thus if H_Q denotes the cone of the adjunction morphism $\eta_{Z,Q}\colon Q \to \mathsf{R}c_{Z,\mathrm{qc},*}\mathsf{L}c_Z^*Q$, then $\mathsf{RHom}_{\mathcal{O}_Z}(P,H_Q) \simeq 0$ for all $P \in \mathsf{D}^-_{\mathrm{pc}}(Z)$. But $\mathsf{D}^-_{\mathrm{pc}}(Z_\lambda)$ contains the perfect complexes of Z, so $H_Q \simeq 0$ [HR17a, Thm. A]. That is, $\eta_{Z,Q}$ is an isomorphism for all $Q \in \mathsf{D}^-_{\mathrm{pc}}(Z)$. Now let $\mathcal{Q} \in \mathsf{D}^-_{\mathrm{pc}}(\mathcal{Z})$. Then $Q \in \mathsf{D}^-_{\mathrm{pc}}(Z)$ and there is an isomorphism $\mathcal{Q} \simeq \mathsf{L}c_Z^*Q$. By what we have proved so far, it follows that $\mathsf{R}c_{Z,\mathrm{qc},*}\mathcal{Q} \simeq \mathsf{R}c_{Z,\mathrm{qc},*}\mathsf{L}c_Z^*Q \simeq Q$. That is, $\mathsf{R}c_{Z,\mathrm{qc},*}$ restricts to a functor from $\mathsf{D}^-_{\mathrm{pc}}(\mathcal{Z})$ to $\mathsf{D}^-_{\mathrm{pc}}(Z)$. It follows immediately from general nonsense that $\mathsf{R}c_{Z,\mathrm{qc},*}$ is right adjoint to $\mathsf{L}c_Z^*\colon \mathsf{D}^-_{\mathrm{pc}}(Z) \to \mathsf{D}^-_{\mathrm{pc}}(\mathcal{Z})$ and we have the claimed adjoint equivalence. □

The following lemma is where the technicalities of this section are buried.

Lemma 5.5. *Assume Setup 5.1. Let $N \in \mathsf{D}^-_{\mathrm{pc}}(X)$ and define*

$$\begin{aligned}\mathcal{V}_N &= \{\mathcal{Q} \in \mathsf{D}(\mathcal{X}) \,:\, \pi_{N,\mathcal{Q}} \text{ is an isomorphism}\},\\ \mathcal{T} &= \{\mathcal{Q} \in \mathsf{D}(\mathcal{X}) \,:\, \epsilon_{\mathcal{Q}} \text{ is an isomorphism}\}, \text{ and}\\ \mathcal{S} &= \{P \in \mathsf{D}_{\mathrm{qc}}(X) \,:\, \eta_P \text{ is an isomorphism}\}.\end{aligned}$$

Then $\langle \mathsf{R}i'_*\mathsf{D}^-_{\mathrm{pc}}(\mathcal{Z})\rangle \subseteq \mathcal{V}_N$. If c and i are tor-independent, then $\langle \mathsf{R}i'_*\mathsf{D}^-_{\mathrm{pc}}(\mathcal{Z})\rangle \subseteq \mathcal{T}$ and $\langle \mathsf{R}i_*\mathsf{D}^-_{\mathrm{pc}}(Z)\rangle \subseteq \mathcal{S}$.

Proof. Clearly, we can view $\mathcal{V}_N$, $\mathcal{T}$, and $\mathcal{S}$ as full subcategories of $\mathsf{D}(\mathcal{X})$, $\mathsf{D}(\mathcal{X})$, and $\mathsf{D}_{\mathrm{qc}}(X)$, respectively. Moreover, they are obviously triangulated and thick subcategories. It remains to prove the following.

(1) If $\mathcal{Q}_0 \in \mathsf{D}^-_{\mathrm{pc}}(\mathcal{Z})$, then $\mathsf{R}i'_*\mathcal{Q}_0 \in \mathcal{V}_N$: this is essentially just the functoriality of the projection formula. Specifically, we apply Lemma A.6 with $\mathcal{C} = \mathsf{D}_{\mathrm{qc}}(X)$, $\mathcal{D} = \mathsf{D}(\mathcal{X})$, $\mathcal{C}' = \mathsf{D}_{\mathrm{qc}}(Z)$, and $\mathcal{D}' = \mathsf{D}(\mathcal{Z})$ with the natural functors and adjoints already described. Lemma B.1 now implies that $\pi_{N,\mathsf{R}i'_*\mathcal{Q}_0}$ is an isomorphism whenever $\pi_{\mathsf{L}i^*N,\mathcal{Q}_0}$ is an isomorphism. But $\mathsf{L}i^*N \in \mathsf{D}^-_{\mathrm{pc}}(Z)$, $\mathcal{Q}_0 \in \mathsf{D}^-_{\mathrm{pc}}(\mathcal{Z})$, and $\mathsf{L}c_Z^*$ and $\mathsf{R}c_{Z,\mathrm{qc},*}$ are an adjoint equivalence on pseudo-coherent complexes (Lemma 5.4), so the claim follows from Remark A.1.

(2) Assume that c and i are tor-independent. If $\mathcal{Q}_0 \in \mathsf{D}^-_{\mathrm{pc}}(\mathcal{Z})$, then $\mathsf{R}i'_*\mathcal{Q}_0 \in \mathcal{T}$: to see this, we note that the following diagram commutes

$$\begin{tikzcd} \mathsf{L}c^*\mathsf{R}c_{\mathrm{qc},*}\mathsf{R}i'_*\mathcal{Q}_0 \arrow[r] \arrow[d, "\epsilon_{\mathsf{R}i'_*\mathcal{Q}_0}"'] & \mathsf{L}c^*\mathsf{R}i_*\mathsf{R}c_{Z,\mathrm{qc},*}\mathcal{Q}_0 \arrow[d] \\ \mathsf{R}i'_*\mathcal{Q}_0 & \mathsf{R}i'_*\mathsf{L}c_Z^*\mathsf{R}c_{Z,\mathrm{qc},*}\mathcal{Q}_0. \arrow[l, "\mathsf{R}i'_*\epsilon_{\mathcal{Q}_0}"'] \end{tikzcd}$$

The claim now follows from functoriality (the top morphism) Lemmas 5.4 (the bottom morphism) and B.2 (the right morphism).

(3) Assume that c and i are tor-independent. If $Q_0 \in \mathsf{D}^-_{\mathrm{pc}}(Z)$, then $\mathsf{R}i_*Q_0 \in \mathcal{S}$: this is almost identical to (2), so is omitted. □

We now introduce a key definition. We appreciate that it is difficult to parse. Such a definition appears necessary, however, to treat the lack of tor-independence that appears in the non-noetherian situation as well as the subtleness of the projection morphism. When tor-independence is available, Proposition 5.7 provides a useful criterion for Z-equivalence.

Definition 5.6. Assume Setup 5.1. Let $\mathcal{M} \in \mathsf{D}(\mathcal{X})$, $\mathcal{N} \in \mathsf{D}^-_{\mathrm{pc}}(\mathcal{X})$. We say that c is

(a) *faithful* along $\mathcal{M}$ at $\mathcal{N}$ if
$$\nu_{\mathcal{M},\mathcal{N}} \colon \mathsf{R}c_{\mathrm{qc},*}\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{R}c_{\mathrm{qc},*}\mathcal{N} \to \mathsf{R}c_{\mathrm{qc},*}(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{N})$$
is an isomorphism; and

(b) an *equivalence* along $\mathcal{M}$ at $\mathcal{N}$ if (a) holds and
$$\mathcal{M} \otimes \epsilon_{\mathcal{N}} \colon \mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathsf{L}c^*\mathsf{R}c_{\mathrm{qc},*}\mathcal{N} \to \mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{N}$$
is an isomorphism.

If the above holds for all $\mathcal{N}$, then we omit the "at $\mathcal{N}$". If $\mathcal{M} = i'_*\mathcal{O}_{\mathcal{Z}}$, then we will replace "$\mathcal{M}$" with "$Z$".

The simplest method to produce an equivalence along Z is to use the following.

Proposition 5.7. *Assume Setup 5.1. Let $M \in \mathsf{D}_{\mathrm{qc}}(X)$ and $\mathcal{N} \in \mathsf{D}^-_{\mathrm{pc}}(\mathcal{X})$.*

(1) *If η_M is an isomorphism, then c is faithful along $\mathsf{L}c^*M$ at $\mathcal{N}$ if and only if $\pi_{M,\mathcal{N}}$ is an isomorphism.*

(2) *If c and i are tor-independent and $M \in \langle \mathsf{R}i_*\mathsf{D}^-_{\mathrm{pc}}(Z)\rangle$, then c is an equivalence along $\mathcal{M}$ at $\mathcal{N}$ if and only if $\pi_{M,\mathcal{N}}$ is an isomorphism.*

Proof. By definition of the projection morphism (A.1), the following diagram commutes:

$$\begin{tikzcd} M \otimes^{\mathsf{L}}_{\mathcal{O}_X} (\mathsf{R}c_{\mathrm{qc},*}\mathcal{N}) \arrow[r, "\eta_M \otimes \mathsf{R}c_{\mathrm{qc},*}\mathcal{N}"] \arrow[dr, "\pi_{M,\mathcal{N}}"'] & \mathsf{R}c_{\mathrm{qc},*}\mathsf{L}c^*M \otimes^{\mathsf{L}}_{\mathcal{O}_X} (\mathsf{R}c_{\mathrm{qc},*}\mathcal{N}) \arrow[d, "\nu_{\mathsf{L}c^*M,\mathcal{N}}"] \\ & \mathsf{R}c_{\mathrm{qc},*}(\mathsf{L}c^*M \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{N}). \end{tikzcd}$$

This proves (1). If $M \in \langle \mathsf{R}i_*\mathsf{D}^-_{\mathrm{pc}}(Z)\rangle$ and c and i are tor-independent, then η_M is an isomorphism (Lemma 5.5). By Lemma A.2, the following diagram commutes:

$$\begin{tikzcd} \mathsf{L}c^*(M \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{R}c_{\mathrm{qc},*}\mathcal{N}) \arrow[r] \arrow[d, "\mathsf{L}c^*\pi_{M,\mathcal{N}}"'] & \mathsf{L}c^*M \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathsf{L}c^*\mathsf{R}c_{\mathrm{qc},*}\mathcal{N} \arrow[d, "\mathsf{L}c^*M \otimes \epsilon_{\mathcal{N}}"] \\ \mathsf{L}c^*\mathsf{R}c_{\mathrm{qc},*}(\mathsf{L}c^*M \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{N}) \arrow[r, "\epsilon_{\mathsf{L}c^*M \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{N}}"] & \mathsf{L}c^*M \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{N}. \end{tikzcd}$$

The top morphism is an isomorphism, as is the bottom (Lemmas 5.5 and B.1). The stated equivalence follows. □

Many examples are provided by the following two results.

Corollary 5.8. *Assume Setup 5.1 and c and i are tor-independent. If*

(1) *i is a Cartier divisor; or*

(2) *$i_*\mathcal{O}_Z$ is perfect; or*

(3) *$\mathcal{O}_{\mathcal{X}}$ is a compact object of $\mathsf{D}(\mathcal{X})$ (see Remark 4.4);*

then c is an equivalence along Z.

Proof. We use the criterion of Proposition 5.7. Case (1) is a special case of (2). In cases (2) and (3) the projection morphism is an isomorphism by Lemma 4.3. □

Corollary 5.9. *Assume Setup 5.1, c and i are tor-independent, and i is a closed immersion. If $\mathcal{O}_X$ is coherent, then there is a perfect complex $M \in \langle \mathsf{R}i_*\mathsf{D}^-_{\mathrm{pc}}(Z)\rangle$ with $\tau^{\geq 0}M \simeq \mathcal{O}_Z$ such that c is an equivalence along $\mathsf{L}c^*M$.*

Proof. By perfect approximation [Stacks, Tag 08HP], there exists a perfect complex $M \in \mathsf{D}^{\leq 0}_{\mathsf{Coh},|Z|}(X)$ with $\tau^{\geq 0}M \simeq \mathcal{O}_Z$. By Lemma 3.4, $M \in \langle \mathsf{R}i_*\mathsf{D}^b_{\mathsf{Coh}}(Z)\rangle$, so $\mathsf{L}c^*M \in \langle \mathsf{R}i'_*\mathsf{D}^-_{\mathrm{pc}}(\mathcal{Z})\rangle$ (Lemma B.2). Now apply Lemma 4.3 and Proposition 5.7. ☐

Lacking tor-independence, these notions can be quite subtle. We will give some interesting examples at the end of this section, however. In the meantime, we content ourselves with the following useful lemma.

Lemma 5.10. *Assume Setup 5.1. Let $\mathcal{M}_0 \in \mathsf{D}^-_{\mathrm{pc}}(\mathcal{Z})$, $\mathcal{N} \in \mathsf{D}^-_{\mathrm{pc}}(\mathcal{X})$. If $\mathsf{R}c_{\mathrm{qc},*}\mathcal{N} \in \mathsf{D}^-_{\mathrm{pc}}(X)$ and c is faithful along $\mathsf{R}i'_*\mathcal{M}_0$ at $\mathcal{N}$, then c is an equivalence along $\mathsf{R}i'_*\mathcal{M}_0$ at $\mathcal{N}$.*

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
\mathsf{R}c_{\mathrm{qc},*}(\mathsf{L}c^*\mathsf{R}c_{\mathrm{qc},*}\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathsf{R}i'_*\mathcal{M}_0) \arrow[r, "\pi_{\mathsf{R}c_{\mathrm{qc},*}\mathcal{N},\mathsf{R}i'_*\mathcal{M}_0}"] \arrow[dr, "\mathsf{R}c_{\mathrm{qc},*}(\epsilon_{\mathcal{N}}\otimes \mathsf{R}i'_*\mathcal{M}_0)"'] & \mathsf{R}c_{\mathrm{qc},*}(\mathcal{N}) \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{R}c_{\mathrm{qc},*}\mathsf{R}i'_*\mathcal{M}_0 \arrow[d, "\nu_{\mathcal{N},\mathsf{R}i'_*\mathcal{M}_0}"] \\
& \mathsf{R}c_{\mathrm{qc},*}(\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathsf{R}i'_*\mathcal{M}_0.)
\end{tikzcd}$$

Since c is faithful along $\mathsf{R}i'_*\mathcal{M}_0$ at $\mathcal{N}$, the vertical map is an isomorphism. By Lemma 5.5, the horizontal map is an isomorphism. It follows that the diagonal map is an isomorphism. Let $\mathcal{Q}$ be a cone for $\epsilon_{\mathcal{N}}$; then $\mathcal{Q} \in \mathsf{D}^-_{\mathrm{pc}}(\mathcal{X})$. Hence, $\mathcal{Q} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathsf{R}i'_*\mathcal{M}_0 \simeq \mathsf{R}i'_*(\mathsf{L}i'^*\mathcal{Q} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{Z}}} \mathcal{M}_0)$ (Lemma B.1). Let $\mathcal{Q}_0 = \mathsf{L}i'^*\mathcal{Q} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{Z}}} \mathcal{M}_0$; then $\mathcal{Q}_0 \in \mathsf{D}^-_{\mathrm{pc}}(\mathcal{Z})$ and

$$0 \simeq \mathsf{R}c_{\mathrm{qc},*}(\mathcal{Q} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathsf{R}i'_*\mathcal{M}_0) \simeq \mathsf{R}c_{\mathrm{qc},*}\mathsf{R}i'_*\mathcal{Q}_0 \simeq \mathsf{R}i'_*\mathsf{R}c_{Z,\mathrm{qc},*}\mathcal{Q}_0.$$

It follows immediately that $\mathcal{Q}_0 \simeq 0$ and the claim follows. ☐

The whole reason for introducing these notions is the following key result.

Proposition 5.11. *Assume Setup 5.1. Let $\mathcal{M} \in \mathsf{D}(\mathcal{X})$ and $\mathcal{N} = \mathsf{L}c^*N$, where $N \in \mathsf{D}^-_{\mathrm{pc}}(X)$. If $\pi_{N,\mathcal{M}}$ is an isomorphism, then the following conditions are equivalent:*

1. *c is faithful along $\mathcal{M}$ at $\mathcal{N}$;*
2. *$\mathsf{R}c_{\mathrm{qc},*}\mathcal{M} \otimes \eta_N$ is an isomorphism.*

Proof. This is immediate from the commutativity of the following diagram (A.1):

$$\begin{tikzcd}
N \otimes^{\mathsf{L}}_{\mathcal{O}_X} (\mathsf{R}c_{\mathrm{qc},*}\mathcal{M}) \arrow[r, "\eta_N \otimes \mathsf{R}c_{\mathrm{qc},*}\mathcal{M}"] \arrow[dr, "\pi_{N,\mathcal{M}}"'] & \mathsf{R}c_{\mathrm{qc},*}\mathsf{L}c^*N \otimes^{\mathsf{L}}_{\mathcal{O}_X} (\mathsf{R}c_{\mathrm{qc},*}\mathcal{M}) \arrow[d, "\nu_{\mathsf{L}c^*N,\mathcal{M}}"] \\
& \mathsf{R}c_{\mathrm{qc},*}(\mathsf{L}c^*N \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{M}).
\end{tikzcd}$$

☐

We conclude this section with more methods to produce examples. These are non-noetherian and not necessarily tor-independent. In particular, they can be safely skipped by those only interested in the proof of Theorem A.

Lemma 5.12. *Assume Setup 5.1.*

(1) *c is faithful along $\mathcal{M}$ if and only if*

$$\mathsf{R}\Gamma(X, \mathsf{R}c_{\mathrm{qc},*}\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{R}c_{\mathrm{qc},*}\mathcal{N}) \to \mathsf{R}\Gamma(\mathcal{X}, \mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{N})$$

is an isomorphism for all $\mathcal{N} \in \mathsf{D}^-_{\mathrm{pc}}(\mathcal{X})$.

(2) *If $f \colon X \to \operatorname{Spec} A$ is flat; $\mathcal{A} = \mathcal{O}_X/f^*I$, where $I \subseteq A$ is an ideal; and*

$$A/I \otimes^{\mathsf{L}}_A \mathsf{R}\Gamma(\mathcal{X}, \mathcal{N}) \to \mathsf{R}\Gamma(\mathcal{X}, \mathsf{L}i'^*\mathcal{N})$$

is an isomorphism for all $\mathcal{N} \in \mathsf{D}^-_{\mathrm{pc}}(\mathcal{X})$; then c is an equivalence along Z.

Proof. For (1): the necessity is clear. For the sufficiency, the perfect complexes compactly generate $\mathsf{D}_{\mathrm{qc}}(X)$ [HR17a, Thm. A], so it suffices to prove that

$$\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(P, \mathsf{R}c_{\mathrm{qc},*}\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{R}c_{\mathrm{qc},*}\mathcal{N}) \to \mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(P, \mathsf{R}c_{\mathrm{qc},*}(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{N}))$$

is an isomorphism for all perfect P. Since perfects are dualizable, the morphism above is an isomorphism if and only if the following morphism is an isomorphism:

$$\mathsf{R}\Gamma(X, P^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{R}c_{\mathrm{qc},*}\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{R}c_{\mathrm{qc},*}\mathcal{N}) \to \mathsf{R}\Gamma(X, P^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{R}c_{\mathrm{qc},*}(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{N})).$$

The projection formula (Lemma 4.3) and adjunction says that this morphism is an isomorphism if and only if the following is an isomorphism:

$$\mathsf{R}\Gamma(X, \mathsf{R}c_{\mathrm{qc},*}(\mathsf{L}c^*P^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{M}) \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{R}c_{\mathrm{qc},*}\mathcal{N})) \to \mathsf{R}\Gamma(\mathcal{X}, \mathsf{L}c^*P^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{N}).$$

The claim follows.

For (2), we take $\mathcal{M} = c^*f^*(A/I) = c^*\mathcal{A} = i'_*\mathcal{O}_{\mathcal{Z}}$; then $\mathsf{R}c_{\mathrm{qc},*}\mathcal{M} \simeq \mathsf{R}i_*\mathsf{R}c_{Z,\mathrm{qc},*}\mathcal{O}_{\mathcal{Z}} \simeq \mathcal{A} = f^*(A/I) \simeq \mathsf{L}f^*(A/I)$. We next observe that the usual projection formula [HR17a, Cor. 4.12] implies that

$$\mathsf{R}\Gamma(X, \mathsf{L}f^*(A/I) \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{R}c_{\mathrm{qc},*}\mathcal{N}) \simeq A/I \otimes^{\mathsf{L}}_A \mathsf{R}\Gamma(X, \mathsf{R}c_{\mathrm{qc},*}\mathcal{N}) \simeq A/I \otimes^{\mathsf{L}}_A \mathsf{R}\Gamma(\mathcal{X}, \mathcal{N}).$$

The claim now follows from (1). □

Remark 5.13. Lemma 5.12 can easily be refined when X is quasi-affine:

(1) c is faithful along $\mathcal{M}$ at $\mathcal{N}$ if and only if the following is an isomorphism:

$$\mathsf{R}\Gamma(X, \mathsf{R}c_{\mathrm{qc},*}\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{R}c_{\mathrm{qc},*}\mathcal{N}) \to \mathsf{R}\Gamma(\mathcal{X}, \mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{N}).$$

(2) If $f \colon X \to \operatorname{Spec} A$ is flat; $\mathcal{A} = \mathcal{O}_X/f^*I$, where $I \subseteq A$ is an ideal; and

$$A/I \otimes^{\mathsf{L}}_A \mathsf{R}\Gamma(\mathcal{X}, \mathcal{N}) \to \mathsf{R}\Gamma(\mathcal{X}, \mathsf{L}i'^*\mathcal{N})$$

is an isomorphism; then c is an equivalence along Z at $\mathcal{N}$.

We have the following non-noetherian and non-tor-independent example that comes from [Stacks, Tag 0DIA].

Example 5.14. Let $\{A_n\}_{n \geq 0}$ be an inverse system of rings with surjective transition maps and locally nilpotent kernel. Let $A = \varprojlim_n A_n$. Let $X \to \operatorname{Spec} A$ be a proper, flat, and finitely presented morphism of algebraic spaces. Let $I_n = \ker(A \to A_n)$ and let $\mathcal{O}_{X_n} = \mathcal{O}_X/I_n\mathcal{O}_{X_n}$. Since $X \to \operatorname{Spec} A$ is flat, the natural map $\mathcal{O}_X \otimes^{\mathsf{L}}_A A_n \to \mathcal{O}_{X_n}$ in $\mathsf{D}(X)$ is an isomorphism. Let $\mathcal{X}$ be the ringed topos with underlying topos $X_{\text{ét}}$ and sheaf of rings $\mathcal{O}_{\mathcal{X}} = \varprojlim_n \mathcal{O}_{X_n}$ in $\mathsf{Mod}(X)$. There is a morphism of ringed topoi $c \colon \mathcal{X} \to X_{\text{ét}}$ corresponding to $\mathcal{O}_X \to \mathcal{O}_{\mathcal{X}}$. Let $n \geq 0$; then [Stacks, Tag 0CQF] and a local calculation shows that $\mathcal{O}_{X_n} \to \mathcal{O}_{\mathcal{X}_n} = \mathcal{O}_{\mathcal{X}}/\ker(\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{X_n})$ is an isomorphism of sheaves of $\mathcal{O}_X$-algebras. Let $i_n \colon X_n \to X$ and $i'_n \colon X_n \to \mathcal{X}$ be the resulting morphisms; note that $c \circ i'_n = i_n$.

First we show that if $\mathcal{M} \in \mathsf{D}^-_{\mathrm{pc}}(\mathcal{X})$, then $\mathsf{R}\Gamma(\mathcal{X},\mathcal{M}) \in \mathsf{D}^-_{\mathrm{pc}}(A)$. For each $n \geq 0$ let $\mathcal{M}_n = \mathsf{L}i'^*_n\mathcal{M} \in \mathsf{D}^-_{\mathrm{pc}}(X_n)$. By [Stacks, Tag 0CQF] and a local calculation, $\mathcal{M} \simeq \operatorname{holim}_n \mathsf{R}i'_{n,*}\mathcal{M}_n$ in $\mathsf{D}(\mathcal{X})$. Also $\mathsf{R}\Gamma(\mathcal{X},-)$ preserves homotopy limits, so

$$\mathsf{R}\Gamma(\mathcal{X},\mathcal{M}) \simeq \operatorname*{holim}_n \mathsf{R}\Gamma(\mathcal{X},\mathsf{R}i'_{n,*}\mathcal{M}_n) \simeq \operatorname*{holim}_n \mathsf{R}\Gamma(X_n,\mathcal{M}_n).$$

Let $M_n = \mathsf{R}\Gamma(X_n,\mathcal{M}_n)$. Then M_n is a pseudo-coherent complex of A_n-modules (a special case of Kiehl's Finiteness Theorem, see [Stacks, Tag 0CSD]) and the projection formula [HR17a, Cor. 4.12] and the flatness of $X \to \operatorname{Spec} A$ implies that:

$$M_{n+1} \otimes^{\mathsf{L}}_{A_{n+1}} A_n \simeq \mathsf{R}\Gamma(X_{n+1},\mathcal{M}_{n+1} \otimes^{\mathsf{L}}_{\mathcal{O}_{X_{n+1}}} \mathcal{O}_{X_n}) \simeq M_n.$$

Thus, M is A-pseudo-coherent and $\mathsf{R}\Gamma(\mathcal{X},\mathcal{M}) \otimes^{\mathsf{L}}_A A_n \simeq \mathsf{R}\Gamma(X_n,\mathcal{M}_n)$ [Stacks, Tag 0CQF]. By Proposition 4.1, $\mathsf{R}c_{\mathrm{qc},*}\mathcal{M} \in \mathsf{D}^-_{\mathrm{pc}}(X)$.

In Setup 5.1, we take $Z = X_0$. By the above and Lemma 5.12(2), c is faithful along Z. By Lemma 5.10, c is even an equivalence along Z.

Example 5.15. Let X be a quasi-compact and quasi-separated algebraic space. Let L be a line bundle on X and let $s \in \Gamma(X,L)$. Let $s^\vee\colon L^\vee \to \mathcal{O}_X$ be the dual morphism and let $I = \operatorname{im}(s^\vee)$ and $K = \ker(s^\vee)$. Let $i\colon Z \hookrightarrow X$ be the closed immersion defined by I. That is, i is the vanishing locus of s. Let $\mathcal{O}_{\mathcal{X}}$ be a sheaf of $\mathcal{O}_X$-algebras (not necessarily quasi-coherent) such that

(a) $\mathcal{O}_X/I \to \mathcal{O}_{\mathcal{X}}/I\mathcal{O}_{\mathcal{X}}$ is an isomorphism; and
(b) $K \cong \ker(s^\vee \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathcal{X}})$.

This holds, for example, when X is locally noetherian, s is a regular section of L (i.e., $K = 0$), and $\mathcal{O}_{\mathcal{X}}$ is the formal completion of X along I. This also holds when X is non-noetherian [Stacks, Tag 0BNG]. More generally, it holds when s is a regular section of L and of c^*L. Note that if s is a regular section, then s remaining a regular section of c^*L is easily seen to be equivalent to the tor-independence of c and i.

Let $\mathcal{A} = \mathcal{O}_X/I$ in Setup 5.1. Let $\mathcal{X} = (X_{\mathrm{\acute{e}t}},\mathcal{O}_{\mathcal{X}})$ and $C = [L^\vee \xrightarrow{s^\vee} \mathcal{O}_X]$, which is perfect. We will establish the following:

(1) η_C is an isomorphism;
(2) c is faithful along $\mathsf{L}c^*C$; and
(3) if $\mathcal{N} \in \mathsf{D}^-_{\mathrm{pc}}(\mathcal{X})$ is such that $\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} C$ or $\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} K$ belongs to $\mathsf{D}_{\mathrm{qc}}(X)$, then c is an equivalence along $\mathsf{L}c^*C$ at $\mathcal{N}$.

Condition (3) is of course trivially satisfied when s is a regular section of L.

We first prove (1). Consider the morphism of distinguished triangles:

$$\begin{tikzcd}
\mathcal{H}^{-1}(C)[1] \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & \mathcal{H}^{-0}(C)[0] \arrow[r] \arrow[d] & \mathcal{H}^{-1}(C)[2] \arrow[d] \\
\mathcal{H}^{-1}(\mathsf{R}c_{\mathrm{qc},}\mathsf{L}c^*C)[1] \arrow[r] & \mathsf{R}c_{\mathrm{qc},}\mathsf{L}c^*C \arrow[r] & \mathcal{H}^0(\mathsf{R}c_{\mathrm{qc},}\mathsf{L}c^*C)[0] \arrow[r] & \mathcal{H}^{-1}(\mathsf{R}c_{\mathrm{qc},}\mathsf{L}c^*C)[2].
\end{tikzcd}$$

Now $\mathcal{H}^{-0}(C) = \mathcal{O}_X/I$ and $\mathcal{H}^{-0}(\mathsf{L}c^*C) = \mathcal{O}_{\mathcal{X}}/I\mathcal{O}_{\mathcal{X}}$, which are isomorphic by (a). Similarly, $\mathcal{H}^{-1}(C) = \ker(L^\vee \to \mathcal{O}_X)$ and $\mathcal{H}^{-1}(\mathsf{L}c^*C) = \ker(L^\vee \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}})$ are isomorphic by (b). In particular, both $\mathcal{H}^0(\mathsf{L}c^*C)$ and $\mathcal{H}^{-1}(\mathsf{L}c^*C)$ are quasi-coherent $\mathcal{O}_X$-modules. It follows immediately that $\mathsf{L}c^*C$ is a quasi-coherent $\mathcal{O}_X$-module and so $\mathsf{R}c_{\mathrm{qc},*}\mathsf{L}c^*C \simeq \mathsf{R}c_*\mathsf{L}c^*C \simeq C \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{O}_{\mathcal{X}}$ as $\mathcal{O}_X$-modules. The claim follows.

We next prove (2). Now condition (a) implies that $\mathsf{Mod}(Z) \simeq \mathsf{Mod}(\mathcal{Z})$, so $\mathsf{D}^-_{\mathrm{pc}}(Z) \simeq \mathsf{D}^-_{\mathrm{pc}}(\mathcal{Z})$. Since C is perfect, Lemma 4.3 and Proposition 5.7(1) together with (1) imply c is faithful along $\mathsf{L}c^*C$. Claim (3) is similar, so its proof is omitted.

The previous example can be generalized to the vanishing locus of a section of vector bundle.

Example 5.16. Let X be a quasi-compact and quasi-separated algebraic space. Let F be a vector bundle on X and let $s \in \Gamma(X, F)$. Let $i\colon Z \hookrightarrow X$ be the vanishing locus of s. Let $\mathcal{O}_{\mathcal{X}}$ be a sheaf of $\mathcal{O}_X$-algebras (not necessarily quasi-coherent) such that $K(s^\vee) \to K(s^\vee) \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathcal{O}_{\mathcal{X}}$ is a quasi-isomorphism of $\mathcal{O}_X$-modules, where $K(s^\vee)$ is the Koszul complex associated to $s^\vee\colon F^\vee \to \mathcal{O}_X$ [FL85, §IV.2]. If s is regular section of F, then this condition is equivalent to s remaining a regular section of $F \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathcal{X}}$. As before, this condition is satisfied when s is a regular section and $\mathcal{O}_{\mathcal{X}}$ is the formal completion of $\mathcal{O}_X$ along $I = \mathrm{im}(s^\vee)$. If F is a line bundle, then $K(s^\vee) = [F^\vee \xrightarrow{s^\vee} \mathcal{O}_X]$. Hence, we see that this condition is equivalent to those in Example 5.15. Arguing as in Example 5.15, one can establish the following:

(1) c is faithful along $\mathsf{L}c^*K(s^\vee)$; and
(2) if $\mathcal{N} \in \mathsf{D}^-_{\mathrm{pc}}(\mathcal{X})$ is such that $\mathcal{N} \otimes^{\mathsf{L}}_{\mathcal{O}_X} K(s^\vee) \in \mathsf{D}_{\mathrm{qc}}(X)$, then c is an equivalence along $\mathsf{L}c^*K(s^\vee)$ at $\mathcal{N}$.

6. Lefschetz Theorems

To illustrate the strength of our reformulation, we can already give a brief proof of the following Lefschetz theorem.

Theorem 6.1. *Let X be a quasi-compact and quasi-separated algebraic space. Let $c\colon \mathcal{X} \to X_{\text{ét}}$ be a morphism of ringed topoi. Let $i\colon Z \to X$ be a closed immersion and let $r \geq 0$ be an integer. If*

(1) *$U = X - Z$ is quasi-affine;*
(2) *$\mathsf{R}\Gamma(X, \mathcal{O}_X) \to \mathsf{R}\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is r-connected; and*
(3) *there exists $\mathcal{M} \in \mathsf{D}(\mathcal{X})$ such that*
   (a) *$\mathsf{R}c_{\mathrm{qc},*}\mathcal{M}$ is perfect with cohomological support $|Z|$, and*
   (b) *c is faithful along $\mathcal{M}$ at $\mathcal{O}_{\mathcal{X}}$;*

then $\mathcal{O}_X \to \mathsf{R}c_{\mathrm{qc},*}\mathcal{O}_{\mathcal{X}}$ is r-connected. In particular, if $E \in \mathsf{Vect}(X)$, then

$$\mathsf{R}\Gamma(X, E) \to \mathsf{R}\Gamma(\mathcal{X}, c^*E)$$

is r-connected.

Proof. By the projection formula (Lemma 4.3), $\pi_{\mathcal{O}_X,\mathcal{M}}$ is an isomorphism. Thus, Proposition 5.11 implies that $\mathsf{R}c_{\mathrm{qc},*}\mathcal{M} \otimes \eta_{\mathcal{O}_X}$ is an isomorphism. Let Q be a cone for $\eta_{\mathcal{O}_X} : \mathcal{O}_X \to \mathsf{R}c_{\mathrm{qc},*}\mathcal{O}_{\mathcal{X}}$; then we have just proved that $Q \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{R}c_{\mathrm{qc},*}\mathcal{M} \simeq 0$. It remains to prove that $\tau^{\leq r} Q \simeq 0$. Let $j : U \to X$ be the resulting open immersion. The theory of smashing Bousfield localizations implies immediately that $Q \simeq \mathsf{R}j_*\mathsf{L}j^*Q$ (e.g., [HR17b, Ex. 1.4]). But U is quasi-affine, so it follows that $\tau^{\leq r}Q \simeq 0$ if and only if $\tau^{\leq r}\mathsf{R}\Gamma(X, Q) \simeq 0$ [HR17a, Cor. 2.8]. Finally, $\mathsf{R}\Gamma(X, \mathcal{O}_X) \to \mathsf{R}\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \simeq \mathsf{R}\Gamma(X, \mathsf{R}c_{\mathrm{qc},*}\mathcal{O}_{\mathcal{X}})$ is r-connected. The result follows. □

We can now prove Theorem B.

Proof of TheoremB. If $i_*\mathcal{O}_Z$ is perfect, then set $\mathcal{M} = \mathsf{L}c^*i_*\mathcal{O}_Z$. If $\mathcal{O}_X$ is coherent, then set $\mathcal{M} = \mathsf{L}c^*M$, where M is as in Lemma 5.9. Since c and i are tor-independent, Corollary 5.8(2) (in the perfect case) and Corollary 5.9 (in the coherent case) imply that c is faithful along $\mathcal{M}$. Moreover, Lemma 5.5 implies that $\mathsf{R}c_{\mathrm{qc},*}\mathcal{M} = \mathsf{R}c_{\mathrm{qc},*}\mathsf{L}c^*M \simeq M$, which is perfect with cohomological support $|Z|$. Now the conditions $\mathrm{H}^0(X,\mathcal{O}_X) \simeq \mathrm{H}^0(\mathcal{X},\mathcal{O}_\mathcal{X})$ and $\mathrm{H}^1(X,\mathcal{O}_X) \hookrightarrow \mathrm{H}^1(\mathcal{X},\mathcal{O}_\mathcal{X})$ are equivalent to $\mathsf{R}\Gamma(X,\mathcal{O}_X) \to \mathsf{R}\Gamma(\mathcal{X},\mathcal{O}_\mathcal{X})$ being 1-connected. The result now follows from Theorem 6.1. □

In the following theorem, we can optimize the above results substantially in the case of a Cartier divisor, making them amenable to an inductive process.

Theorem 6.2. *Let X be a quasi-compact and quasi-separated algebraic space. Let $c\colon \mathcal{X} \to X_{\text{ét}}$ be a morphism of ringed topoi. Let L be a line bundle on X, $s \in \Gamma(X,L)$, and $i\colon Z \hookrightarrow X$ its vanishing locus. Let $r \geq 0$ be an integer. If*

(1) *$U = X - Z$ is quasi-affine;*

(2) *$C \to \mathsf{R}c_{\mathrm{qc},*}\mathsf{L}c^*C$ is r-connected, where $C = [L^\vee \xrightarrow{s^\vee} \mathcal{O}_X]$; and*

(3) *$\mathsf{R}\Gamma(X,\mathcal{O}_X) \to \mathsf{R}\Gamma(\mathcal{X},\mathcal{O}_\mathcal{X})$ is r-connected;*

then $\mathcal{O}_X \to \mathsf{R}c_{\mathrm{qc},*}\mathcal{O}_\mathcal{X}$ is r-connected. In particular, if $E \in \mathsf{Vect}(X)$, then

$$\mathsf{R}\Gamma(X,E) \to \mathsf{R}\Gamma(\mathcal{X},c^*E)$$

is r-connected. In addition, if s is a regular section of L and c^*L (equivalently, c and i are tor-independent), then we may replace (2) *with*

(2') *$\mathcal{O}_Z \to \mathsf{R}c'_{\mathrm{qc},*}\mathcal{O}_\mathcal{Z}$ is r-connected, where $c'\colon \mathcal{Z} = (\mathcal{X}, c^*\mathcal{O}_Z) \to (X_{\text{ét}},\mathcal{O}_Z)$.*

Proof. If $N \in \mathsf{D}_{\mathrm{qc}}(X)$, let Q_N be a cone for η_N; then we must show that $\tau^{\leq r}Q_{\mathcal{O}_X} \simeq 0$. Clearly C is perfect, so by Lemma 4.3, $\pi_{C,\mathsf{L}c^*\mathcal{O}_X}$ is an isomorphism. Hence, the commutative diagram (A.3) implies that $Q_{\mathcal{O}_X} \otimes^{\mathsf{L}}_{\mathcal{O}_X} C \simeq Q_C$. By condition (2), we conclude that $\tau^{\leq r}(Q_{\mathcal{O}_X} \otimes^{\mathsf{L}}_{\mathcal{O}_X} C) \simeq 0$. We now let $Q = Q_{\mathcal{O}_X}$ and consider the distinguished triangle:

$$L^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_X} Q \longrightarrow Q \longrightarrow C \otimes^{\mathsf{L}}_{\mathcal{O}_X} Q \longrightarrow L^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_X} Q[1].$$

But $L^\vee$ is a line bundle, so $\tau^{\leq r}(L^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_X} Q) \simeq L^\vee \otimes^{\mathsf{L}}_{\mathcal{O}_X} \tau^{\leq r}Q$. It follows immediately from the distinguished triangle above that $C \otimes^{\mathsf{L}}_{\mathcal{O}_X} \tau^{\leq r}Q \simeq 0$. Let $j\colon U = X - Z \hookrightarrow X$ be the resulting open immersion; then the theory of smashing Bousfield localizations implies immediately that $\tau^{\leq r}Q \simeq \mathsf{R}j_*\mathsf{L}j^*\tau^{\leq r}Q$ (e.g., [HR17b, Ex. 1.4]). But U is quasi-affine, so it follows that $\tau^{\leq r}Q \simeq 0$ if and only if $\mathsf{R}\Gamma(X,\tau^{\leq r}Q) \simeq 0$ [HR17a, Cor. 2.8]. Finally, $\mathsf{R}\Gamma(X,\mathcal{O}_X) \to \mathsf{R}\Gamma(\mathcal{X},\mathcal{O}_\mathcal{X}) \simeq \mathsf{R}\Gamma(X,\mathsf{R}c_{\mathrm{qc},*}\mathcal{O}_\mathcal{X})$ is r-connected (by (3)), so $0 \simeq \tau^{\leq r}\mathsf{R}\Gamma(X,Q) \simeq \mathsf{R}\Gamma(X,\tau^{\leq r}Q)$. The result follows. □

7. Pseudo-conservation

Let $\mathcal{X}$ be a ringed topos. We say that a collection $S \subseteq \mathsf{D}(\mathcal{X})$ is *pseudo-conservative* if whenever $\mathcal{M} \in \mathsf{D}^-_{\mathrm{pc}}(\mathcal{X})$ satisfies $\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_\mathcal{X}} \mathcal{Q} \simeq 0$ for all $\mathcal{Q} \in S$, then $\mathcal{M} \simeq 0$.

Example 7.1. Let X be a quasi-compact and quasi-separated algebraic space. Let $|X|_{\mathrm{cl}}$ be the set of closed points of X. The collection $\{\kappa(x)\}_{x\in|X|_{\mathrm{cl}}}$ is pseudo-conservative. This is immediate from Nakayama's Lemma.

Example 7.2. Let X be a locally ringed space. The collection $\{\kappa(x)\}_{x\in|X|_{\mathrm{cl}}}$ is pseudo-conservative. This is again immediate from Nakayama's Lemma.

Example 7.3. Let A be a ring. Let $X \to \operatorname{Spec} A$ be a quasi-compact and closed morphism of algebraic spaces. Let $I \subseteq A$ be an ideal contained in the Jacobson radical of A. Let $X_0 = X \times_{\operatorname{Spec} A} \operatorname{Spec}(A/I)$ and take $i\colon X_0 \to X$ be the resulting closed immersion. Then $\{\mathcal{O}_{X_0}\}$ is pseudo-conservative. Indeed, if $M \in \mathsf{D}^-_{\mathrm{pc}}(X)$ is non-zero, then its top cohomology group $\mathcal{H}^{top}(M)$ is finitely generated. It follows that its support W is a non-empty closed subset of X. The image of W in $\operatorname{Spec} A$ is closed and non-empty so must meet $\operatorname{Spec}(A/I)$ because I is contained in the Jacobson radical of A.

We have the following useful lemma.

Lemma 7.4. *Let $\mathcal{X}$ be a ringed topos, where $\mathcal{O}_{\mathcal{X}}$ is coherent. Let $S \subseteq \mathsf{D}^-(\mathcal{X})$ be a collection of objects. Consider the collection:*

$$S' = \{\mathcal{H}^{t(\mathcal{Q})}(\mathcal{Q}) \,:\, \mathcal{Q} \in S\} \subseteq \mathsf{Mod}(\mathcal{X}),$$

where $t(\mathcal{Q})$ denotes the top cohomological degree of $\mathcal{Q}$. The following are equivalent:

(1) *S is pseudo-conservative;*
(2) *if $\mathcal{M} \in \mathsf{Coh}(\mathcal{X})$ and $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{Q} \cong 0$ for all $\mathcal{Q} \in S'$, then $\mathcal{M} \cong 0$.*

Proof. This is immediate from the following: if $\mathcal{M}$, $\mathcal{N} \in \mathsf{D}^-(\mathcal{X})$, then

$$\mathcal{H}^{t(\mathcal{M})+t(\mathcal{Q})}(\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{N}) \cong \mathcal{H}^{t(\mathcal{M})}(\mathcal{M}) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{H}^{t(\mathcal{N})}(\mathcal{N}).$$

□

8. GAGA

In this section, we prove our general GAGA theorem. We will see in §9 that this implies all existing results in the literature for algebraic spaces. Given what we have already established, its proof is straightforward.

Theorem 8.1. *Let X be a quasi-compact and quasi-separated algebraic space. Let $c\colon \mathcal{X} \to X_{\text{ét}}$ be a morphism of ringed topoi. Consider a family of quasi-coherent sheaves of $\mathcal{O}_X$-algebras $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$. Let Z_λ and $\mathcal{Z}_\lambda$ be the ringed topoi $(X_{\text{ét}}, \mathcal{A}_\lambda)$ and $(\mathcal{X}, c^*\mathcal{A}_\lambda)$, respectively. There is an induced 2-commutative diagram of ringed topoi:*

$$\begin{tikzcd} \mathcal{Z}_\lambda \arrow[r, "c_\lambda"] \arrow[d, "i'_\lambda"'] & Z_\lambda \arrow[d, "i_\lambda"] \\ \mathcal{X} \arrow[r, "c"] & X_{\text{ét}}. \end{tikzcd}$$

For each $\lambda \in \Lambda$, let $\mathcal{M}_\lambda \in \langle \mathsf{R}i'_{\lambda,*}\mathsf{D}^-_{\mathrm{pc}}(\mathcal{Z}_\lambda)\rangle$.

(i) *Let $N \in \mathsf{D}^-_{\mathrm{pc}}(X)$. Assume that $\mathsf{R}c_{\mathrm{qc},*}\mathsf{L}c^*N \in \mathsf{D}^-_{\mathrm{pc}}(X)$ and $\{\mathsf{R}c_{\mathrm{qc},*}\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$ is pseudo-conservative. If for all $\lambda \in \Lambda$, c is faithful along $\mathcal{M}_\lambda$ at $\mathsf{L}c^*N$, then*
$$\eta_M\colon M \to \mathsf{R}c_{\mathrm{qc},*}\mathsf{L}c^*M$$
is an isomorphism.

(ii) *Let $\mathcal{N} \in \mathsf{D}^-_{\mathrm{pc}}(\mathcal{X})$. Assume that $\mathsf{R}c_{\mathrm{qc},*}\mathcal{N} \in \mathsf{D}^-_{\mathrm{pc}}(X)$ and $\{\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$ is pseudo-conservative. If for all $\lambda \in \Lambda$, c is an equivalence along $\mathcal{M}_\lambda$ at $\mathcal{N}$, then*
$$\epsilon_{\mathcal{M}}\colon \mathsf{L}c^*\mathsf{R}c_{\mathrm{qc},*}\mathcal{M} \to \mathcal{M}$$
is an isomorphism.

In addition assume that $* \in \{b, -\}$ and

(1) *X is proper and pseudo-coherent over an affine scheme $\operatorname{Spec} A$;*

(2) $\mathsf{R}\Gamma(\mathcal{X},-)$ *sends* $\mathsf{D}^*_{\mathrm{pc}}(\mathcal{X})$ *to* $\mathsf{D}^*_{\mathrm{pc}}(A)$*; and*
(3) *if* $*=b$*, then* $\mathsf{L}c^*$ *sends* $\mathsf{D}^b_{\mathrm{pc}}(X)$ *to* $\mathsf{D}^b_{\mathrm{pc}}(\mathcal{X})$*.*

If $\{\mathsf{R}c_{\mathrm{qc},*}\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$ *(resp.* $\{\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$*) is pseudo-conservative and for all* $\lambda\in\Lambda$*,* c *is faithful (resp. an equivalence) by* $\mathcal{M}_\lambda$*, then*

$$\mathsf{L}c^*\colon \mathsf{D}^*_{\mathrm{pc}}(X)\to\mathsf{D}^*_{\mathrm{pc}}(\mathcal{X})$$

is fully faithful (resp. essentially surjective).

Proof. For (i), by Lemma 5.5 and Proposition 5.11, we have that $\mathsf{R}c_{\mathrm{qc},*}\mathcal{M}_\lambda\otimes\eta_N$ is an isomorphism for all $\lambda\in\Lambda$. But N and $\mathsf{R}c_{\mathrm{qc},*}\mathsf{L}c^*N\in\mathsf{D}^-_{\mathrm{pc}}(X)$, so $\mathrm{cone}(\eta_N)\in\mathsf{D}^-_{\mathrm{pc}}(X)$. Since $\{\mathsf{R}c_{\mathrm{qc},*}\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$ is pseudo-conservative, the claim follows.

For (ii), $\mathcal{M}_\lambda\otimes\epsilon_{\mathcal{N}}$ is an isomorphism for all $\lambda\in\Lambda$. But $\mathcal{N}$, $\mathsf{L}c^*\mathsf{R}c_{\mathrm{qc},*}\mathcal{N}\in\mathsf{D}^-_{\mathrm{pc}}(\mathcal{X})$, so $\mathrm{cone}(\epsilon_{\mathcal{N}})\in\mathsf{D}^-_{\mathrm{pc}}(\mathcal{X})$. Since $\{\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$ is pseudo-conservative, the claim follows.

The last claim is immediate from the above and Proposition 4.1. □

9. Applications

We will now demonstate how Theorem 8.1 establishes all existing GAGA results in the literature. We begin with the following tor-independent GAGA theorem.

Theorem 9.1. *Let* A *be a ring. Let* $\pi\colon X\to\operatorname{Spec}A$ *be a proper and pseudo-coherent morphism of algebraic spaces. Let* $c\colon\mathcal{X}\to X_{\text{ét}}$ *be a morphism of ringed topoi. Consider a family of quasi-coherent sheaves of ideals* $\{I_\lambda\subseteq\mathcal{O}_X\}_{\lambda\in\Lambda}$*. Let* Z_λ *and* $\mathcal{Z}_\lambda$ *be the ringed topoi* $(X_{\text{ét}},\mathcal{O}_X/I_\lambda)$ *and* $(\mathcal{X},c^*\mathcal{O}_X/I_\lambda)$*, respectively. There is an induced 2-commutative diagram of ringed topoi*

$$\begin{tikzcd} \mathcal{Z}_\lambda \arrow[r, "c_\lambda"] \arrow[d, "i'_\lambda"'] & Z_\lambda \arrow[d, "i_\lambda"] \\ \mathcal{X} \arrow[r, "c"] & X_{\text{ét}}. \end{tikzcd}$$

Let $*\in\{b,-\}$*. Assume that*
(a) $\mathsf{R}\Gamma(\mathcal{X},-)$ *sends* $\mathsf{D}^*_{\mathrm{pc}}(\mathcal{X})$ *to* $\mathsf{D}^*_{\mathrm{pc}}(A)$*;*
(b) *for all* $\lambda\in\Lambda$*,* c *and* i_λ *are tor-independent;*
(c) *for all* $\lambda\in\Lambda$*,* $\mathsf{L}c^*_\lambda\colon\mathsf{D}^-_{\mathrm{pc}}(Z_\lambda)\to\mathsf{D}^-_{\mathrm{pc}}(\mathcal{Z}_\lambda)$ *is an equivalence;*
(d) *if* $*=b$*, then* $\mathsf{L}c^*$ *sends* $\mathsf{D}^b_{\mathrm{pc}}(X)$ *to* $\mathsf{D}^b_{\mathrm{pc}}(\mathcal{X})$*;*
(e) $\mathcal{O}_X$ *is coherent; or* $\mathcal{O}_{\mathcal{X}}$ *is a compact object of* $\mathsf{D}(\mathcal{X})$*; or for all* $\lambda\in\Lambda$*,* $i_{\lambda,*}\mathcal{O}_{Z_\lambda}$ *is perfect.*

If $\{\mathcal{O}_{Z_\lambda}\}_{\lambda\in\Lambda}$ *(resp.* $\{\mathcal{O}_{\mathcal{Z}_\lambda}\}_{\lambda\in\Lambda}$*) is pseudo-conservative, then*

$$\mathsf{L}c^*\colon \mathsf{D}^*_{\mathrm{pc}}(X)\to\mathsf{D}^*_{\mathrm{pc}}(\mathcal{X})$$

is fully faithful (resp. essentially surjective).

Proof. In the case where $\mathcal{O}_{\mathcal{X}}$ is a compact object of $\mathsf{D}(\mathcal{X})$ or $i_{\lambda,*}\mathcal{O}_{Z_\lambda}$ is perfect for all $\lambda\in\Lambda$, the result is immediate from Theorem 8.1 and Corollary 5.8. In the case where $\mathcal{O}_X$ is coherent, we apply Corollary 5.9 to produce a perfect complex $M_\lambda\in\langle\mathsf{R}i_*\mathsf{D}^-_{\mathrm{pc}}(Z_\lambda)\rangle$ with c an equivalence along $\mathsf{L}c^*M_\lambda$ for all $\lambda\in\Lambda$. But if $\{\mathcal{O}_{Z_\lambda}\}_{\lambda\in\Lambda}$ (resp. $\{\mathcal{O}_{\mathcal{Z}_\lambda}\}_{\lambda\in\Lambda}$) is pseudo-conservative, then $\{M_\lambda\}_{\lambda\in\Lambda}$ (resp. $\{\mathsf{L}c^*M_\lambda\}_{\lambda\in\Lambda}$) is pseudo-conservative (Lemma 7.4). Now apply Theorem 8.1. □

We now prove Theorem A.

Proof of Theorem A. We simply verify the conditions (a)–(e) of Theorem 9.1, with Λ indexing the set of closed points of X and $* = b$. Clearly, conditions (1) and (2) of Theorem A imply condition (a). Condition (e) is obvious.

Let $S = \mathcal{X}_{\mathrm{cl},c}$. We next check conditions (b) and (d). Let $M \in \mathsf{D}^-_{\mathsf{Coh}}(X)$ and $s \in S$; then $(\mathsf{L}c^*M)_s \simeq M_{c(s)} \otimes^{\mathsf{L}}_{\mathcal{O}_{X,c(s)}} \mathcal{O}_{\mathcal{X},s}$. If $\tau^{<n}M \simeq 0$ for some n, then $(\tau^{<n}\mathsf{L}c^*M)_s \simeq \tau^{<n}(M_{c(s)} \otimes^{\mathsf{L}}_{\mathcal{O}_{X,c(s)}} \mathcal{O}_{\mathcal{X},s})$. By condition (5), $\tau^{<n}(M_{c(s)} \otimes^{\mathsf{L}}_{\mathcal{O}_{X,c(s)}} \mathcal{O}_{\mathcal{X},s}) \simeq 0$. Condition (4) gives $\tau^{<n}(\mathsf{L}c^*M) \simeq 0$, so $\mathsf{L}c^* \colon \mathsf{D}^-_{\mathsf{Coh}}(X) \to \mathsf{D}^-_{\mathsf{Coh}}(\mathcal{X})$ is t-exact.

We next prove that if $s \in S$, then the induced map on residue fields $\kappa(c(s)) \to \kappa(s)$ is an isomorphism. By condition (5), $\kappa(c(s)) \to \kappa(c(s)) \otimes_{\mathcal{O}_{X,c(s)}} \mathcal{O}_{\mathcal{X},s}$ is an isomorphism. But we also have a surjective morphism $\kappa(c(s)) \otimes_{\mathcal{O}_{X,c(s)}} \mathcal{O}_{\mathcal{X},s} \to \kappa(s)$. Consequently, the map of fields $\kappa(c(s)) \to \kappa(s)$ is surjective, hence an isomorphism.

We next prove that if $s \in S$, then the natural morphism $\epsilon \colon c^*\kappa(c(s)) \to k(s)$ is an isomorphism of sheaves. If $t \in \mathcal{X}$, then $(c^*\kappa(c(s)))_t = \kappa(c(s))_{c(t)} \otimes_{\mathcal{O}_{X,c(t)}} \mathcal{O}_{\mathcal{X},t}$. Condition (3) tells us that this is 0 if $t \neq s$ and is $\kappa(c(s)) \otimes_{\mathcal{O}_{X,c(s)}} \mathcal{O}_{\mathcal{X},s} \cong \kappa(s)$ if $t = s$. Observe that condition (3) implies that $c^{-1}(X - \{c(s)\}) = \mathcal{X} - \{s\}$ and so $\mathcal{X} - \{s\}$ is admissible and $c^*\kappa(c(s))$ is the sheafification of the presheaf $\mathcal{C} \colon \mathcal{U} \mapsto \varinjlim_{W \supseteq c(\mathcal{U})}(\kappa(c(s))(W) \otimes_{\mathcal{O}_X(W)} \mathcal{O}_{\mathcal{X}}(\mathcal{U}))$; obviously, $\mathcal{C}(\mathcal{U}) = 0$ if $s \notin \mathcal{U}$ and the same is true for its sheafification. We now apply the elementary Lemma 9.2 to the sheaf $c^*\kappa(c(s))$ with the point s to obtain the isomorphism $c^*\kappa(c(s)) \to \kappa(s)$.

Let $s \in S$ and let $c_s \colon (\mathcal{X}, \kappa(s)) \to (|X|, \kappa(c(s)))$ be the induced morphism of G-ringed spaces. Then $c_s^* \colon \mathsf{Mod}(|X|, \kappa(c(s))) \to \mathsf{Mod}(\mathcal{X}, \kappa(s))$ is an equivalence (Lemma 9.2). Hence, condition (c) is satisfied and we have a t-exact equivalence

$$\mathsf{L}c^* \colon \mathsf{D}^b_{\mathsf{Coh}}(X) \to \mathsf{D}^b_{\mathsf{Coh}}(\mathcal{X})$$

The cohomological comparison and the equivalence on hearts follows trivially. □

Lemma 9.2. *Let $\mathcal{X}$ be a G-ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{\mathcal{X}}$-modules. Let $x \in \mathcal{X}$ be a closed point and $i \colon \{x\} \to \mathcal{X}$ the inclusion. We will regard $\{x\}$ as a ringed space with structure sheaf $\mathcal{O}_{\mathcal{X},x}$. Assume that:*

1. *if $\mathcal{W} \subseteq \mathcal{X}$ is admissible and $x \notin \mathcal{W}$, then $\mathcal{F}(\mathcal{W}) = 0$; and*
2. *$\mathcal{X} - \{x\}$ has an admissible cover.*

Then the natural morphism $\mathcal{F} \to i_*i^*\mathcal{F}$ is an isomorphism. In particular, the functor

$$\mathsf{Mod}(\mathcal{X}, \kappa(x)) \to \mathsf{Mod}(\kappa(x)) \colon \mathcal{G} \mapsto \mathcal{G}_x$$

is an exact equivalence of abelian categories.

Proof. We first prove that $\mathcal{F} \to i_*i^*\mathcal{F}$ is an isomorphism of sheaves of $\mathcal{O}_X$-modules. That it is an epimorphism of sheaves is obvious. That it is a monomorphism: condition (1) implies that it suffices to prove that if $\mathcal{U}$ is admissible and $x \in \mathcal{U}$, then the natural morphism $\mathcal{F}(\mathcal{U}) \to \mathcal{F}_x$ is injective. Let $f \in \mathcal{F}(\mathcal{U})$ and suppose that $f_x = 0$ in $\mathcal{F}_x$. By definition of the stalk, there exists an admissible open $\mathcal{V}$ of $\mathcal{U}$ containing x such that $f|_{\mathcal{V}} = 0$ in $\mathcal{F}(\mathcal{V})$. By condition (2), $\mathcal{X} - \{x\} = \cup_{i \in I}\mathcal{W}_i$, where each $\mathcal{W}_i$ is admissible and I is some indexing set. Now $\mathcal{F}$ is a sheaf, so $\mathcal{F}(\mathcal{U}) \subseteq \mathcal{F}(\mathcal{V}) \times \prod_{i \in I}\mathcal{F}(\mathcal{U} \cap \mathcal{W}_i) = \mathcal{F}(\mathcal{V})$, since $x \notin \mathcal{U} \cap \mathcal{W}_i$ for all i. Since $f|_{\mathcal{V}} = 0$, the claim follows.

To prove the equivalence, we may assume that $\mathcal{O}_{\mathcal{X}} = \kappa(x)$. Obviously, if $\mathcal{G}$ is a sheaf of $\kappa(x)$-modules, then $\mathcal{G}$ satisfies condition (1); hence, $\mathcal{G} \to i_*i^*\mathcal{G}$ is an isomorphism. This proves that the functor $\mathcal{G} \mapsto \mathcal{G}_x$ is fully faithful. The essential surjectivity is trivial and the result follows. □

Remark 9.3. We now explain the necessity of the conditions in Theorem A. Assume that $\mathcal{X}$ is a G-locally ringed space with $\mathcal{O}_{\mathcal{X}}$ coherent and there is a set of closed points S of $\mathcal{X}$ such that

(a) if $s \in S$, then $\mathcal{O}_{\mathcal{X},s}$ is noetherian;
(b) if $s \in S$, then $\kappa(s) \in \mathsf{Coh}(\mathcal{X})$; and
(c) if $\mathcal{F} \in \mathsf{Coh}(\mathcal{F})$ and $\mathcal{F}_s = 0$ for all $s \in S$, then $\mathcal{F} = 0$.

Now suppose that $c\colon \mathcal{X} \to X$ is a morphism of G-locally ringed spaces such that $\mathsf{L}c^*\colon \mathsf{D}^-_{\mathsf{Coh}}(X) \to \mathsf{D}^-_{\mathsf{Coh}}(\mathcal{X})$ is a t-exact equivalence. We claim that $S = c^{-1}(X_{\mathrm{cl}})$ and the remaining conditions of Theorem A are satisfied. Conditions (1) and (2) are obvious. Let $x \in X_{\mathrm{cl}}$; then $\Gamma(\mathcal{X}, c^*\kappa(x)) = \Gamma(X, \kappa(x)) \neq 0$. Hence, there exists $s \in S$ such that $(c^*\kappa(x))_s \neq 0$. Now if $t \in S$, then $(c^*\kappa(x))_t = \kappa(x)_{c(t)} \otimes_{\mathcal{O}_{X,c(t)}} \mathcal{O}_{\mathcal{X},t}$. We deduce immediately that $c(s) = x$. Thus, we have a local ring homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{\mathcal{X},s}$ that induces a morphism on residue fields $\kappa(x) \to \kappa(s)$ and the induced morphism of sheaves of $\mathcal{O}_{\mathcal{X}}$-algebras $c^*\kappa(x) \to \kappa(s)$ is surjective. But then $\kappa(x) \to c_{\mathrm{qc},*}\kappa(s)$ is a surjective morphism of coherent $\mathcal{O}_X$-modules and $\kappa(x)$ is a skyscraper sheaf, so this forces the morphism to be an isomorphism. Hence, $c^*\kappa(x) \simeq \kappa(s)$. Examining the stalk at s, we further see that $\kappa(x) \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{\mathcal{X},s} \to \kappa(s)$ is an isomorphism. Also, comparison of global sections of $c^*\kappa(x)$ and $\kappa(s)$ shows that the morphism of fields $\kappa(x) \to \kappa(s)$ is an isomorphism. Further, if $\exists\, t \in \mathcal{X} - \{s\}$ such that $c(t) = x$, then $(c^*\kappa(x))_t \neq 0$; but this is nonsense in view of the established isomorphism $c^*\kappa(x) \simeq \kappa(s)$. Also if $s \in S$, then $(c_{\mathrm{qc},*}\kappa(s))_x \neq 0$ for some $x \in X_{\mathrm{cl}}$. Hence, there exists a non-zero morphism $c_{\mathrm{qc},*}\kappa(s) \to \kappa(x)$ and so a non-zero morphism $\kappa(s) \to c^*\kappa(x)$. It follows that $c(s) = x$; thus, $S = c^{-1}(X_{\mathrm{cl}})$ and $c\colon S \to X_{\mathrm{cl}}$ is bijective. Finally, the t-exactness of $\mathsf{L}c^*\colon \mathsf{D}^-_{\mathsf{Coh}}(X) \to \mathsf{D}^-_{\mathsf{Coh}}(\mathcal{X})$ implies that $\mathrm{Tor}_1^{\mathcal{O}_{X,c(s)}}(\kappa(c(s)), \mathcal{O}_{\mathcal{X},s}) = 0$ whenever $s \in S$. By the local criterion for flatness, $\mathcal{O}_{X,c(s)} \to \mathcal{O}_{\mathcal{X},s}$ is flat whenever $s \in S$. This establishes conditions (3), (4), and (5) of Theorem A.

It is easy to use Theorems A, 8.1, and 9.1 to prove existing GAGA results.

Example 9.4 (Analytic spaces)**.** Let $X \to \operatorname{Spec}\mathbb{C}$ be a proper scheme. Let $c\colon X_{\mathrm{an}} \to X$ be its complex analytification. Now X_{an} is a Hausdorff topological space and c is bijective on closed points; indeed $|X_{\mathrm{an}}| = X(\operatorname{Spec}\mathbb{C})$. Also, the local rings of $\mathcal{O}_{X_{\mathrm{an}}}$ are noetherian and the induced morphism $\mathcal{O}_{X,c(x)} \to \mathcal{O}_{X_{\mathrm{an}},x}$ is an isomorphism on maximal-adic completions [SGA1, XII.1.1]. By Remark 1.1, we see that conditions (3), (4), and (5) are satisfied. The Grauert–Remmert Theorem [GR84, 10.5.6] implies that if $\mathcal{F} \in \mathsf{Coh}(X_{\mathrm{an}})$, then condition (2) is satisfied. Oka's Coherence Theorem [GR84, 2.5.3] is that $\mathcal{O}_{X_{\mathrm{an}}}$ is coherent, so condition (1) is satisfied. By Theorem A we may conclude that if $F \in \mathsf{Coh}(X)$, then

$$\mathrm{H}^i(X, F) \simeq \mathrm{H}^i(X_{\mathrm{an}}, F_{\mathrm{an}})$$

and $c^*\colon \mathsf{Coh}(X) \to \mathsf{Coh}(X_{\mathrm{an}})$ is an equivalence.

Example 9.5 (Formal GAGA)**.** Let $X \to \operatorname{Spec} R$ be a proper morphism of schemes. Assume that R is noetherian. Let $I \subseteq R$ be an ideal and assume that R is complete with respect to the I-adic topology. Let $c\colon \hat{X} \to X$ be the formal completion of X along the closed subscheme $X_0 = X \otimes_R (R/I)$. It is easily verified using the results of [EGA, III_1] that c satisfies the hypotheses of Theorem A. Hence, we have the cohomological comparison result and the equivalence on categories of sheaves. It is also easy to use these arguments and Theorem 9.1 to prove formal GAGA for proper algebraic spaces. We again leave this as an exercise to the reader. One can also use these arguments to prove the formal GAGA statements of [FK13], which hold for certain non-noetherian base rings A (e.g., A is the a-adic completion of a finitely presented V-algebra, where V is an a-adically complete valuation ring.).

Example 9.6 (Rigid GAGA)**.** Let $X \to \operatorname{Spec} R$ be a proper morphism of schemes. Let k be a complete nonarchimean field. Assume that R is an affinoid k-algebra; that is, it is a Banach k-algebra that is a quotient of some Tate algebra $T_n = k\langle\langle Y_1, \dots, Y_n\rangle\rangle$, where Y_n is the subalgebra of $k[[Y_1, \dots, Y_n]]$ consisting of power series that are convergent with respect to the Gauss norm (i.e., suprema of coefficients). Associated to X is a natural morphism of G-locally ringed spaces $c\colon X_{\mathrm{rig}} \to X$, where X_{rig} is a rigid analytic space. The underlying topological space of X_{rig} is Hausdorff and its points correspond to closed points of X. Moreover, $\mathcal{O}_{X_{\mathrm{rig}}}$ is a coherent sheaf with noetherian local rings. Also, R is noetherian. Kiehl's Finiteness Theorem [Kie67] implies that the cohomology of coherent sheaves on $\mathcal{O}_{X_{\mathrm{rig}}}$ satisfies the condition (2) of Theorem A. Again, we get the cohomological comparison result and equivalence on categories of coherent sheaves. Using [CT09], one can make sense of rigid analytifications of separated algebraic spaces. This allows one to prove rigid GAGA in this context too. One can also prove adic and Berkovich GAGA statements using this method.

Example 9.7 (Non-noetherian formal GAGA)**.** Here we will use Theorem 8.1 to prove the GAGA result in the Stacks Project [Stacks, Tag 0DIA]. The situation is as in Example 5.14, and it is immediate from 8.1 that we obtain an equivalence

$$\mathsf{L}c^*\colon \mathsf{D}^-_{\mathrm{pc}}(X) \to \mathsf{D}^-_{\mathrm{pc}}(\mathcal{X}).$$

We now use Theorem 9.1 to prove a generalization of *relative* analytic GAGA from projective n-space to proper algebraic spaces, which recently appeared in [AT18, Thm. C.1.1]. As far as we are aware, this was previously unknown.

Example 9.8. Let $(\mathcal{Y}, Y)$ be an *analytic germ* [AT18, App. B]; that is, $\mathcal{Y}$ is an analytic space and $i\colon Y \subseteq \mathcal{Y}$ is a semianalytic subset. Associated to $(\mathcal{Y}, Y)$ is a locally ringed space, $\mathcal{Y}_Y$, which has underlying topological space Y and sheaf of rings $i^{-1}\mathcal{O}_{\mathcal{Y}}$. In particular, $\mathcal{Y}_Y$ only depends on an open neighborhood of Y in $\mathcal{Y}$ and its structure sheaf is easily checked to be coherent using Oka's Theorem.

A morphism of germs $(\mathcal{Y}_1, Y_1) \to (\mathcal{Y}_2, Y_2)$ is a morphism of analytic spaces $f\colon \mathcal{U}_1 \to \mathcal{Y}_2$, where $\mathcal{U}_1 \subseteq \mathcal{Y}_1$ is an open neigborhood of Y_1 and $f(Y_1) \subseteq Y_2$—we call f a *representative*. Morphisms of germs are declared equivalent if they admit equal representatives. The morphism is *without boundary* if there is a representative f with $f^{-1}(Y_2) = Y_1$. Obviously, a morphism of germs gives rise to a well-defined morphism on their locally ringed spaces. A $\mathcal{Y}_Y$-space is a morphism of germs $(\mathcal{X}, X) \to (\mathcal{Y}, Y)$.

If P is a property of morphisms of morphisms of analytic spaces, then we say that a morphism of germs is P if there exists a representative that is P and without boundary. By the usual Grauert–Remmert Theorem and Proper Base Change in topology, it follows that a proper morphism of germs sends a bounded complex of coherent sheaves to a bounded complex of coherent sheaves.

A germ is *affinoid* if it admits a closed immersion into a germ of the form $(\mathbb{C}^n, D)$, where D is a closed polydisc in $\mathbb{C}^n$. If $(\mathcal{Y}, Y)$ is affinoid, then (i) $R = \Gamma(\mathcal{Y}_Y, \mathcal{O}_{\mathcal{Y}_Y})$ is an excellent noetherian $\mathbb{C}$-algebra; (ii) $\Gamma(Y, -)$ induces an exact equivalence between coherent $\mathcal{O}_{\mathcal{Y}_Y}$-modules and finitely generated R-modules; the map $c\colon |Y| \to |\operatorname{Spec} R|$ is injective with image the closed points; and if $y \in Y$, then $\mathcal{O}_{\operatorname{Spec} R, c(y)} \to \mathcal{O}_{Y,y}$ is a morphism of noetherian local rings that induces an isomorphism on maximal adic completions [AT18, Lem. B.6.1].

If X is a locally of finite type and locally separated algebraic space over $\operatorname{Spec} R$, then there is a relative analytification $X_{(\mathcal{Y},Y)_{\mathrm{an}}}$. More precisely, there is a functor from locally of finite type and locally separated algebraic spaces over Spec R to $\mathcal{Y}_Y$-spaces. It is easily verifed that separated, proper morphisms are sent to separated and proper morphisms, respectively. Similarly, étale morphisms are sent to local isomorphisms. It is easily verified that there is an induced flat morphism of ringed sites $c\colon X_{(\mathcal{Y},Y)_{\mathrm{an}},\text{ét}} \to X_{\text{ét}}$. We claim that the induced functor:

$$\mathsf{D}^-_{\mathsf{Coh}}(X) \to \mathsf{D}^-_{\mathsf{Coh}}(X_{(\mathcal{Y},Y)_{\mathrm{an}}})$$

is a t-exact equivalence. This generalizes the main result of [AT18, App. C] from $X = \mathbb{P}^n_R$ to algebraic spaces that are proper over $\operatorname{Spec} R$. To do this, we simply verify the conditions of Theorem 9.1, with Λ the set of closed points of X. In light of the above discussion and the arguments in Example 9.4, this is obvious.

The following is a variant of the results established in [BJ14, §1]. It is a simple consequence of Theorem 6.2.

Theorem 9.9. *Let X be a quasi-affine scheme. Let $A = \Gamma(X, \mathcal{O}_X)$ and $a \in A$. Let $X_0 = \operatorname{Spec}(A/a) \cap X \subseteq X$ and let $c\colon \hat{X} \to X$ be the formal completion of X along X_0. Assume*

(1) *A is a-adically complete;*
(2) *a is not a zero divisor of A; and*
(3) *$\bigcap_{n\geq 0} a^n \mathrm{H}^1(X, \mathcal{O}_X) = 0$ (see Remark 9.10).*

Then

$$c^*\colon \mathsf{Vect}(X) \to \mathsf{Vect}(\hat{X})$$

is fully faithful.

Proof. By Theorem 6.2 and Example 5.15, it suffices to prove $\mathrm{H}^0(X, \mathcal{O}_X) \simeq \mathrm{H}^0(\hat{X}, \mathcal{O}_{\hat{X}})$ and $\mathrm{H}^1(X, \mathcal{O}_X) \hookrightarrow \mathrm{H}^1(\hat{X}, \mathcal{O}_{\hat{X}})$. Since a is not a zero divisor of A, we can apply the second exact sequence of [BJ14, Lem. 1.2] to every affine open of X to conclude that $\mathcal{O}_{\hat{X}} \simeq \operatorname{holim}_n \mathcal{O}_X/a^n\mathcal{O}_X$ in $\mathsf{D}(X)$. Hence, we may apply [*loc. cit.*] again to obtain injections:

$$\varprojlim_n \mathrm{H}^0(X, \mathcal{O}_X)/a^n \hookrightarrow \mathrm{H}^0(\hat{X}, \mathcal{O}_{\hat{X}}) \quad \text{and} \quad \varprojlim_n \mathrm{H}^1(X, \mathcal{O}_X)/a^n \hookrightarrow \mathrm{H}^1(\hat{X}, \mathcal{O}_{\hat{X}}).$$

Since $\mathrm{H}^0(X, \mathcal{O}_X) = A$ and is a-adically complete, the first map is an isomorphism; that is, $\mathrm{H}^0(X, \mathcal{O}_X) \simeq \mathrm{H}^0(\hat{X}, \mathcal{O}_{\hat{X}})$. But the kernel of the map $\mathrm{H}^1(X, \mathcal{O}_X) \to \varprojlim_n \mathrm{H}^1(X, \mathcal{O}_X)/a^n$ is just $\bigcap_{n\geq 0} a^n \mathrm{H}^1(X, \mathcal{O}_X)$, which vanishes by hypothesis. Hence, $\mathrm{H}^1(X, \mathcal{O}_X) \hookrightarrow \mathrm{H}^1(\hat{X}, \mathcal{O}_{\hat{X}})$ too. □

Remark 9.10. Condition (3) of Theorem 9.9, while subtle, is frequently met in practice. Indeed, if $\mathrm{H}^1(X, \mathcal{O}_X)$ is a finitely generated A-module, then it holds: we always have $\mathrm{H}^1(X, \mathcal{O}_X)_a = 0$, so if $\mathrm{H}^1(X, \mathcal{O}_X)$ is a finitely generated A-module, then $a^N \mathrm{H}^1(X, \mathcal{O}_X) = 0$ for all $N \gg 0$.

Example 9.11. Let $(A, \mathfrak{m})$ be a noetherian local ring. Let $X = \operatorname{Spec} A - \{\mathfrak{m}\}$. Let $a \in A$ and let $c\colon \hat{X} \to X$ be the a-adic completion of X (as a formal scheme). If

(1) A is a-adically complete,
(2) a is not a zero divisor of A, and
(3) $\operatorname{depth}_{\mathfrak{m}}(A) \geq 3$,

then $c^* \colon \mathsf{Vect}(X) \to \mathsf{Vect}(\hat{X})$ is fully faithful.

Appendix A. The projection formula

We recall some results on the projection formula. An excellent source is [FHM03]. Let $(\mathcal{C}, \otimes, \alpha, \sigma, 1, \lambda, \rho)$ be a symmetric monoidal category. That is,

- $\mathcal{C}$ is a category;
- $- \otimes - \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor;
- for each triple x, y, $z \in \mathcal{C}$ there is a functorial isomorphism
$$\alpha_{x,y,z} \colon (x \otimes y) \otimes z \simeq x \otimes (y \otimes z),$$
which satisfies the pentagram law;
- for each pair x, $y \in \mathcal{C}$ there is a functorial isomorphism
$$\sigma_{x,y} \colon x \otimes y \simeq y \otimes x$$
such that $\sigma_{y,x} \circ \sigma_{x,y} = \mathrm{Id}$, and is compatible with α in the obvious sense;
- $1 \in \mathcal{C}$;
- for each $x \in \mathcal{C}$ there are functorial isomorphisms
$$\lambda_x \colon 1 \otimes x \simeq x \quad \text{and} \quad \rho_x \colon x \otimes 1 \simeq x,$$
such that $\lambda_1 = \rho_1$ and are compatible with α and σ in the obvious sense.

Typically, we will just denote this data by $\mathcal{C}$. For background material on symmetric monoidal categories, we refer the interested reader to [SR72, ML98]. Consider an adjoint pair of functors:
$$L \colon \mathcal{C} \rightleftarrows \mathcal{D} \colon R,$$
where $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal categories, and L is strong monoidal. This means that for c_1, $c_2 \in \mathcal{C}$, there is a natural isomorphism
$$\mu_{c_1,c_2} \colon L(c_1 \otimes_{\mathcal{C}} c_2) \simeq L(c_1) \otimes_{\mathcal{D}} L(c_2)$$
as well as an isomorphism
$$\iota \colon L(1_{\mathcal{C}}) \simeq 1_{\mathcal{D}}$$
that are compatible with the rest of the data defining a symmetric monoidal category. We denote this package of data by (L, R, μ, ι).

If $c \in \mathcal{C}$ and $d \in \mathcal{D}$, then we have the resulting unit/counit morphisms
$$\eta_c \colon c \to RL(c) \quad \text{and} \quad \epsilon_d \colon LR(d) \to d.$$
If d_1, $d_2 \in \mathcal{D}$, then there is a natural "conjugate" of μ_{c_1,c_2},
$$\nu_{d_1,d_2} \colon R(d_1) \otimes_{\mathcal{C}} R(d_2) \to R(d_1 \otimes_{\mathcal{D}} d_2).$$
It is obtained as the adjoint to the composition:
$$L(R(d_1) \otimes_{\mathcal{C}} R(d_2)) \xrightarrow{L(\mu_{R(d_1),R(d_2)})} LR(d_1) \otimes_{\mathcal{D}} LR(d_2) \xrightarrow{\epsilon_{d_1} \otimes \epsilon_{d_2}} d_1 \otimes_{\mathcal{D}} d_2.$$
In perhaps more familiar terms: the right adjoint to a strong monoidal functor is lax-monoidal. It follows that if $c \in \mathcal{C}$ and $d \in \mathcal{D}$, then there is a natural *projection morphism*
$$\pi_{c,d} \colon c \otimes_{\mathcal{C}} R(d) \to R(L(c) \otimes_{\mathcal{D}} d).$$

Indeed, it is given as the composition:

$$c \otimes_{\mathcal{C}} R(d) \xrightarrow{\eta_c \otimes \mathrm{Id}_{R(d)}} RL(c) \otimes_{\mathcal{C}} R(d) \xrightarrow{n_{L(c),d}} R(L(c) \otimes_{\mathcal{D}} d). \tag{A.1}$$

Remark A.1. Note that if L is an equivalence, then $\pi_{c,d}$ is an isomorphism.

There is another way to produce a projection morphism

$$\tilde{\pi}_{c,d} \colon c \otimes_{\mathcal{C}} R(d) \to R(L(c) \otimes_{\mathcal{D}} d).$$

It can be given as the adjoint to the composition:

$$L(c \otimes_{\mathcal{C}} R(d)) \xrightarrow{\mu_{c,R(d)}} L(c) \otimes_{\mathcal{D}} LR(d) \xrightarrow{\mathrm{Id}_{L(c)} \otimes \epsilon_d} L(c) \otimes_{\mathcal{D}} d.$$

We wish to point out that ν (and so consequently π) depend on the choice of the right adjoint R. Occasionally, it will be useful to observe this, and we do so by using a suitable superscript (e.g., $\nu^{L,R}$).

Lemma A.2. $\pi_{c,d} = \tilde{\pi}_{c,d}$.

Proof. The adjoint to $\pi_{c,d}$ factors as:

$$L(c \otimes_{\mathcal{C}} R(d)) \xrightarrow{L(\eta_c \otimes \mathrm{Id})} L(RL(c) \otimes_{\mathcal{C}} R(d)) \xrightarrow{L(\mu_{RL(c),R(d)})} LRL(c) \otimes_{\mathcal{D}} LR(d) \xrightarrow{\epsilon_{L(c)} \otimes \epsilon_d} L(c) \otimes_{\mathcal{D}} d.$$

The following square also commutes, by naturality:

$$\begin{tikzcd}
L(c \otimes_{\mathcal{C}} R(d)) \arrow[r, "L(\eta_c \otimes \mathrm{Id})"] \arrow[d, "\mu_{c,R(d)}"'] & L(RL(c) \otimes_{\mathcal{C}} R(d)) \arrow[d, "L(\mu_{RL(c),R(d)})"] \\
L(c) \otimes_{\mathcal{D}} LR(d) \arrow[r, "L(\eta_c) \otimes \mathrm{Id}"] & LRL(c) \otimes_{\mathcal{D}} LR(d).
\end{tikzcd}$$

Hence, the adjoint to $\pi_{c,d}$ factors as:

$$L(c \otimes_{\mathcal{C}} R(d)) \xrightarrow{\mu_{c,R(d)}} L(c) \otimes_{\mathcal{D}} LR(d) \xrightarrow{L(\eta_c) \otimes \mathrm{Id}} LRL(c) \otimes_{\mathcal{D}} LR(d) \xrightarrow{\epsilon_{L(c)} \otimes \epsilon_d} L(c) \otimes_{\mathcal{D}} d.$$

By the unit/conuit equations for adjunction, the composition of the last two morphisms results in $\mathrm{Id} \otimes \epsilon_d$. The result now follows. $\square$

Remark A.3. If $\kappa \colon (L, R, \mu^L, \iota^L) \Rightarrow (L', R', \mu', \iota')$ is a natural transformation of symmetric monoidal functors, then there is a canonically induced natural transformation $\kappa^\vee \colon R' \Rightarrow R$ between their right adjoints. If $c \in \mathcal{C}$ and $d \in \mathcal{D}$, then it is easily verified from Lemma A.2 that the following diagram commutes:

$$\begin{tikzcd}
c \otimes_{\mathcal{C}} R'(d) \arrow[r, "\pi^{L',R'}_{c,d}"] \arrow[d, "\mathrm{Id} \otimes \kappa^\vee"'] & R'(L'(c) \otimes_{\mathcal{D}} d) \arrow[dr, "\kappa^\vee"] & \\
c \otimes_{\mathcal{C}} R(d) \arrow[r, "\pi^{L,R}_{c,d}"'] & R(L(c) \otimes_{\mathcal{D}} d) \arrow[r, "R(\kappa)"'] & R(L'(c) \otimes_{\mathcal{D}} d).
\end{tikzcd}$$

In the following lemma we record some useful commutative diagrams.

Lemma A.4. *Let c, $x \in \mathcal{C}$ and $d \in \mathcal{D}$. The following diagrams commute.*

(A.2)
$$\begin{tikzcd} L(c \otimes_{\mathcal{C}} R(d)) \arrow[r, "\mu_{c,R(d)}"] \arrow[d, "L(\pi_{c,d})"'] & L(c) \otimes_{\mathcal{D}} LR(d) \arrow[d, "\mathrm{Id} \otimes \epsilon_d"] \\ LR(L(c) \otimes d) \arrow[r, "\epsilon_{L(c)\otimes d}"] & L(c) \otimes_{\mathcal{D}} d. \end{tikzcd}$$

(A.3)
$$\begin{tikzcd} c \otimes_{\mathcal{C}} x \arrow[r, "\eta_{c\otimes x}"] \arrow[d, "c\otimes\eta_x"'] & RL(c \otimes_{\mathcal{C}} x) \arrow[d, "R(\mu_{c,x})"] \\ c \otimes_{\mathcal{C}} RL(x) \arrow[r, "\pi_{c,L(x)}"'] & R(L(c) \otimes_{\mathcal{D}} L(x)). \end{tikzcd}$$

Proof. The diagram (A.2) is just a restatement of Lemma A.2. For the commutativity of (A.3), we observe that the adjoint of the composition going right and then down is simply $\mu_{c,x}$. Working the other way, we see that the adjoint map is the composition:

$$L(c \otimes_{\mathcal{C}} x) \xrightarrow{L(\mathrm{Id}\otimes\eta_x)} L(c \otimes_{\mathcal{C}} RL(x)) \xrightarrow{\mu_{c,RL(x)}} L(c) \otimes_{\mathcal{D}} LRL(x) \xrightarrow{\mathrm{Id}\otimes\epsilon_{L(x)}} L(c) \otimes_{\mathcal{D}} L(x).$$

By functoriality and naturality, the following diagram commutes:

$$\begin{tikzcd} L(c \otimes_{\mathcal{C}} x) \arrow[r, "\mu_{c,x}"] \arrow[d, "L(\mathrm{Id}\otimes\eta_x)"'] & L(c) \otimes_{\mathcal{D}} L(x) \arrow[d, "\mathrm{Id}\otimes L\eta_x"] \\ L(c \otimes_{\mathcal{C}} RL(x)) \arrow[r, "\mu_{c,RL(x)}"] & L(c) \otimes_{\mathcal{D}} LRL(x). \end{tikzcd}$$

It follows that the map we are interested in is actually the composition:

$$L(c \otimes_{\mathcal{C}} x) \xrightarrow{\mu_{c,x}} L(c) \otimes_{\mathcal{D}} L(x) \xrightarrow{\mathrm{Id}\otimes L\eta_x} L(c) \otimes_{\mathcal{D}} LRL(x) \xrightarrow{\mathrm{Id}\otimes\epsilon_{L(x)}} L(c) \otimes_{\mathcal{D}} L(x).$$

By the unit/counit equations for the adjunction, the final two morphisms compose to give the identity. The result follows. ☐

The following two lemmas establish the functoriality properties of the projection morphism.

Lemma A.5. *Consider symmetric monoidal functors:*

$$\mathcal{C} \xrightarrow{(L,R,\mu^L,\iota^L)} \mathcal{D} \xrightarrow{(S,T,\mu^S,\iota^S)} \mathcal{D}'.$$

Then RT is right adjoint to SL and

(1) *$SL\colon \mathcal{C} \to \mathcal{D}'$ is also symmetric monoidal, via the compositions:*

$$\mu^{SL}_{c_1,c_2}\colon SL(c_1 \otimes_{\mathcal{C}} c_2) \xrightarrow{S(\mu^L_{c_1,c_2})} S(L(c_1) \otimes_{\mathcal{D}} L(c_2)) \xrightarrow{\mu^S_{L(c_1),L(c_2)}} SL(c_1) \otimes_{\mathcal{D}'} SL(c_2)$$

$$\iota^{SL}\colon SL(1_{\mathcal{C}}) \xrightarrow{S(\iota^L)} S(1_{\mathcal{D}}) \xrightarrow{\iota^S} 1_{\mathcal{D}'}.$$

(2) *The conjugate (via RT) to $\mu^{SL}_{c_1,c_2}$ is the composition:*

$$\nu^{SL,RT}_{d'_1,d'_2}\colon RT(d'_1)\otimes_{\mathcal{C}} RT(d'_2) \xrightarrow{\nu^{L,R}_{T(d'_1),T(d'_2)}} R(T(d'_1)\otimes_{\mathcal{D}} T(d'_2)) \xrightarrow{R(\nu^{S,T}_{d'_1,d'_2})} RT(d'_1\otimes_{\mathcal{D}'} d'_2).$$

(3) *If $c \in \mathcal{C}$ and $d' \in \mathcal{D}'$, then the following diagram commutes:*

$$\begin{tikzcd} c \otimes_{\mathcal{C}} RT(d') \arrow[r, "\pi^{L,R}_{c,T(d')}"] \arrow[dr, "\pi^{SL,RT}_{c,d'}"'] & R(L(c) \otimes_{\mathcal{D}} T(d')) \arrow[d, "R(\pi^{S,T}_{Lc,d'})"] \\ & RT(SL(c) \otimes_{\mathcal{D}'} d'). \end{tikzcd}$$

Proof. Claim (1) is straightforward. Claim (2) follows from (1). Claim (3) follows from the definition of the projection morphism (A.1) and (2). □

Lemma A.6. *Consider a 2-commutative diagram of symmetric monoidal categories:*

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "L'"] & \mathcal{D}' \\ \mathcal{C} \arrow[u, "F"] \arrow[r, "L"'] \arrow[ur, phantom, "\Searrow\kappa"] & \mathcal{D} \arrow[u, "S"'] \end{tikzcd}$$

Assume that L, L', F, and S admit respective right adjoints R, R', G, and T. Let $c \in \mathcal{C}$ and $d \in \mathcal{D}'$. Then the following diagram commutes:

$$\begin{tikzcd} c \otimes_{\mathcal{C}} RT(d) \arrow[r, "\mathrm{Id}\otimes\kappa^\vee", "\sim"'] \arrow[d, "\pi_{c,T(d)}"'] & c \otimes_{\mathcal{C}} GR'(d) \arrow[r, "\pi_{c,R'(d)}"] & G(F(c) \otimes_{\mathcal{C}'} R'(d)) \arrow[d, "G\pi_{F(c),d}"] \\ R(L(c) \otimes_{\mathcal{D}} T(d)) \arrow[r, "R\pi_{L(c),d}"] & RT(SL(c) \otimes_{\mathcal{D}'} d) \arrow[r, "\kappa^\vee(\kappa\otimes\mathrm{Id})", "\sim"'] & GR'(L'F(c) \otimes_{\mathcal{D}'} d) \end{tikzcd}$$

Proof. Combine Lemma A.5(3) with Remark A.3. □

An object $c \in \mathcal{C}$ is *dualizable* if there is a triple (c^*, s, t), where $c^* \in \mathcal{C}$ and $s\colon 1 \to c \otimes c^*$ and $t\colon c^* \otimes c \to 1$ are morphisms such that the two compositions

$$c \xrightarrow{\lambda_c^{-1}} 1 \otimes c \xrightarrow{s\otimes\mathrm{Id}} (c \otimes c^*) \otimes c \xrightarrow{\alpha_{c,c^*,c}} c \otimes (c^* \otimes c) \xrightarrow{\mathrm{Id}\otimes t} c \otimes 1_{\mathcal{C}} \xrightarrow{\rho_c} c,$$

$$c^* \xrightarrow{\rho_{c^*}^{-1}} c^* \otimes 1 \xrightarrow{\mathrm{Id}\otimes s} c^* \otimes (c \otimes c^*) \xrightarrow{\alpha_{c^*,c,c^*}} (c^* \otimes c) \otimes c^* \xrightarrow{t\otimes\mathrm{Id}} 1 \otimes c^* \xrightarrow{\lambda_{c^*}} c^*$$

are the identity morphism. Another way of expressing this is that the functor $c^* \otimes -$ is left adjoint to $c \otimes -$. In the following standard lemma, we do not require the existence of a right adjoint R to L.

Lemma A.7. *If c is dualizable, then $L(c)$ is dualizable. More precisely: let (c^*, s, t) be a dual of c. Then $(L(c^*), s_L, t_L)$, where s_L is the composition:*

$$1_{\mathcal{D}} \xrightarrow{e^{-1}} L(1_{\mathcal{C}}) \xrightarrow{L(s)} L(c \otimes_{\mathcal{C}} c^*) \xrightarrow{\mu_{c,c^*}} L(c) \otimes_{\mathcal{D}} L(c^*),$$

and t_L is the composition:

$$L(c^*) \otimes_{\mathcal{D}} L(c) \xrightarrow{\mu^{-1}_{c^*,c}} L(c^* \otimes_{\mathcal{C}} c) \xrightarrow{L(t)} L(1_{\mathcal{C}}) \xrightarrow{e} 1_{\mathcal{D}}.$$

is dual to $L(c)$.

Proof. This is a routine diagram chase. □

We now come to the main result of this appendix.

Theorem A.8. *If $c \in \mathcal{C}$ is dualizable, then*

$$\pi_{c,d} \colon c \otimes_{\mathcal{C}} R(d) \to R(L(c) \otimes_{\mathcal{D}} d)$$

is an isomorphism.

It is not difficult to prove that $c \otimes_{\mathcal{C}} R(d)$ and $R(L(c) \otimes_{\mathcal{D}} d)$ are isomorphic when c is dualizable. The subtlety is showing that this isomorphism can be witnessed by the projection morphism $\pi_{c,d}$. In applications, this is critical.

The standard reference for Theorem A.8 (in the context of *closed* symmetric monoidal categories) is [FHM03, Prop. 3.12]. Note that Theorem A.8 is not actually proved in [*loc. cit.*]—there is an extra coherence condition for strong monoidal functors specified in [FHM03, Eq. 3.7]. It is shown in [MS06, Rem. 2.2.10], however, that this coherence condition is implied by the other conditions. Because of its importance to this article, we give a self-contained proof here using dualizables.

Proof of Theorem A.8. Let (c^*, s, t) be a dual of c. Let $x \in \mathcal{C}$. Then observe that we have the following natural sequence of bijections:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(x, c \otimes_{\mathcal{C}} R(d)) &\cong \mathrm{Hom}_{\mathcal{C}}(c^* \otimes_{\mathcal{C}} x, R(d)) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(L(c^* \otimes_{\mathcal{C}} x), d) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(L(c^*) \otimes_{\mathcal{D}} L(x), d) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(L(x), L(c) \otimes_{\mathcal{D}} d) \qquad \text{(Lemma A.7)} \\
&\cong \mathrm{Hom}_{\mathcal{C}}(x, R(L(c) \otimes_{\mathcal{D}} d)).
\end{aligned}$$

By the Yoneda lemma, it follows that there is a unique isomorphism

$$\pi'_{c,d} \colon c \otimes R(d) \simeq R(L(c) \otimes_{\mathcal{D}} d)$$

inducing the above. By Lemma A.2, it remains to prove that $\tilde{\pi}_{c,d} = \pi'_{c,d}$. We will do this using the Yoneda lemma. Fix $f \colon x \to c \otimes_{\mathcal{C}} R(d)$. By definition, the L-R adjoint to the composition $\tilde{\pi}_{c,d} \circ f$ we can express as the composition:

$$L(x) \xrightarrow{L(f)} L(c \otimes_{\mathcal{C}} R(d)) \xrightarrow{\mu_{c,R(d)}} L(c) \otimes_{\mathcal{D}} LR(d) \xrightarrow{\mathrm{Id} \otimes \epsilon_d} L(c) \otimes_{\mathcal{D}} d.$$

The adjoint to this morphism (afforded by $L(c^*) \otimes -$ and $L(c) \otimes -$) is thus the composition:

$$\begin{aligned}
L(c^*) \otimes_{\mathcal{D}} L(x) &\xrightarrow{\mathrm{Id} \otimes L(f)} L(c^*) \otimes_{\mathcal{D}} L(c \otimes_{\mathcal{C}} R(d)) \\
&\xrightarrow{\mathrm{Id} \otimes \mu_{c,R(d)}} L(c^*) \otimes_{\mathcal{D}} (L(c) \otimes_{\mathcal{D}} LR(d)) \\
&\xrightarrow{\alpha_{\mathcal{D},L(c^*),L(c),LR(d)}} (L(c^*) \otimes_{\mathcal{D}} L(c)) \otimes_{\mathcal{D}} LR(d) \\
&\xrightarrow{\mathrm{Id} \otimes \epsilon_d} (L(c^*) \otimes_{\mathcal{D}} L(c)) \otimes_{\mathcal{D}} d \xrightarrow{t_L \otimes \mathrm{Id}} 1_{\mathcal{D}} \otimes_{\mathcal{D}} d \xrightarrow{\lambda_{\mathcal{D},d}} d.
\end{aligned}$$

We now look at the image of f under the compositions defining $\pi'_{c,d}$ via the Yoneda lemma. What we see is that:

$$
\begin{aligned}
f \mapsto (c^* \otimes_{\mathcal{C}} x &\xrightarrow{\mathrm{Id}\otimes f} c^* \otimes_{\mathcal{C}} (c \otimes_{\mathcal{C}} R(d)) \xrightarrow{\alpha_{\mathcal{C},c^*,c,R(d)}} (c^* \otimes_{\mathcal{C}} c) \otimes_{\mathcal{D}} R(d) \\
&\xrightarrow{t\otimes\mathrm{Id}} 1_{\mathcal{C}} \otimes_{\mathcal{C}} R(d) \xrightarrow{\lambda_{\mathcal{C},R(d)}} R(d)) \\
\mapsto (L(c^* \otimes_{\mathcal{C}} x) &\xrightarrow{L(\mathrm{Id}\otimes f)} L(c^* \otimes_{\mathcal{C}} (c \otimes_{\mathcal{C}} R(d))) \\
&\xrightarrow{L(a_{\mathcal{C},c^*,c,R(d)})} L((c^* \otimes_{\mathcal{C}} c) \otimes_{\mathcal{C}} R(d)) \xrightarrow{L(t\otimes\mathrm{Id})} L(1_{\mathcal{C}} \otimes_{\mathcal{C}} R(d)) \\
&\xrightarrow{L(\lambda_{R(d)})} LR(d) \xrightarrow{\epsilon_d} d)
\end{aligned}
$$

It remains to show that precomposing the above morphism with $\mu^{-1}_{c^*,x}$ coincides with the other morphism described above. This follows from the commutativity of the following diagram, and that all of the vertical arrows are isomorphisms:

$$
\begin{tikzcd}
L(c^* \otimes x) \arrow[r, "\mathrm{Id}\otimes L(f)"] \arrow[d, "\mu_{c^*,x}"'] & L(c^* \otimes (c \otimes Rd)) \arrow[r, "\mathrm{Id}\otimes\mu_{c,Rd}"] \arrow[d, "\mu_{c^*,c\otimes Rd}"'] & L((c^* \otimes c) \otimes Rd) \arrow[r, "\alpha_{\mathcal{D}}"] & L(c^* \otimes c) \otimes LRd \arrow[r, "L(t)\otimes\mathrm{Id}"] \arrow[d, "\mu_{c^*,c}\otimes\mathrm{Id}"'] & L(1_{\mathcal{C}}) \otimes LRd \arrow[d, "\iota\otimes\mathrm{Id}"] \\
L(c^*) \otimes L(x) \arrow[r, "\mathrm{Id}\otimes L(f)"'] & L(c^*) \otimes L(c \otimes Rd) \arrow[r, "\mathrm{Id}\otimes\mu_{c,Rd}"] & L(c^*) \otimes (Lc \otimes LRd) \arrow[r, "L(\alpha_{\mathcal{C}})"] & (L(c^*) \otimes Lc) \otimes LRd \arrow[r, "t_L\otimes\mathrm{Id}"] & 1_{\mathcal{D}} \otimes LRd.
\end{tikzcd}
$$

□

Appendix B. Two lemmas for ringed topoi

We include in this appendix two simple lemmas, which we expect to be well-known to experts.

Lemma B.1. *Let $\mathcal{W}$ be a ringed topos. Let $\mathcal{B}$ be a sheaf of $\mathcal{O}_{\mathcal{W}}$-algebras. Let $\mathcal{W}'$ be the ringed topos $(\mathcal{W}, \mathcal{B})$. There is an induced morphism of ringed topoi $j\colon \mathcal{W}' \to \mathcal{W}$. Let $\mathcal{M} \in \mathsf{D}(\mathcal{W})$ and $\mathcal{N} \in \mathsf{D}(\mathcal{W}')$. Then the projection morphism*

$$\pi_{\mathcal{M},\mathcal{N}}\colon \mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{W}}} \mathsf{R}j_*\mathcal{N} \to \mathsf{R}j_*(\mathsf{L}j^*\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{W}'}} \mathcal{N})$$

is an isomorphism. In particular, if $\mathcal{Q} \in \mathsf{D}^-_{\mathrm{pc}}(\mathcal{W})$ and $\mathcal{P} \in \langle \mathsf{R}j_*\mathsf{D}^-_{\mathrm{pc}}(\mathcal{W}')\rangle$, then $\mathcal{Q} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{W}}} \mathcal{P} \in \langle \mathsf{R}j_*\mathsf{D}^-_{\mathrm{pc}}(\mathcal{W}')\rangle$.

Proof. Let $\mathcal{F}$ be a K-flat complex of $\mathcal{O}_{\mathcal{W}}$-modules quasi-isomorphic to $\mathcal{M}$ and $\mathcal{P}$ a K-flat complex of $\mathcal{O}_{\mathcal{W}'}$-modules quasi-isomorphic to $\mathcal{N}$. The exactness of j_* implies that $\mathcal{M} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{W}}} \mathsf{R}j_*\mathcal{N}$ is the total complex of $(\mathcal{F}^r \otimes_{\mathcal{O}_{\mathcal{W}}} j_*\mathcal{P}^s)_{r,s}$. Clearly,

$$\mathcal{F}^r \otimes_{\mathcal{O}_{\mathcal{W}}} j_*\mathcal{P}^s = j_*(j^*\mathcal{F}^r \otimes_{\mathcal{O}_{\mathcal{W}'}} \mathcal{P}^s).$$

Moreover, j_* commutes with the formation of total complexes (it commutes with small coproducts). The result is now immediate. For the latter claim, we set

$$\mathsf{D}_{\mathcal{Q}} = \{\mathcal{R} \in \langle \mathsf{R}j_*\mathsf{D}^-_{\mathrm{pc}}(\mathcal{W}')\rangle \,:\, \mathcal{Q} \otimes^{\mathsf{L}} \mathcal{R} \in \langle \mathsf{R}j_*\mathsf{D}^-_{\mathrm{pc}}(\mathcal{W}')\rangle\}.$$

Clearly, $\mathsf{D}_{\mathcal{Q}}$ is a thick triangulated subcategory of $\langle \mathsf{R}j_*\mathsf{D}^-_{\mathrm{pc}}(\mathcal{W}')\rangle$. Thus, it suffices to prove that $\mathcal{N} \in \mathsf{D}^-_{\mathrm{pc}}(\mathcal{W}')$ implies $\mathcal{Q} \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathcal{W}}} \mathsf{R}j_*\mathcal{N} \in \langle \mathsf{R}j_*\mathsf{D}^-_{\mathrm{pc}}(\mathcal{W}')\rangle$. This is obvious from the projection formula. □

Lemma B.2. *Let $\pi\colon \mathcal{Y} \to \mathcal{W}$ be a morphism of ringed topoi. Let $\mathcal{B}$ be a sheaf of $\mathcal{O}_{\mathcal{W}}$-algebras. Let $\mathcal{W}'$ and $\mathcal{Y}'$ be the ringed topoi $(\mathcal{W}, \mathcal{B})$ and $(\mathcal{Y}, \pi^*\mathcal{B})$, respectively. There is an induced 2-commutative diagram of ringed topoi:*

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "\pi'"] \arrow[d, "j'"'] & \mathcal{W}' \arrow[d, "j"] \\ \mathcal{Y} \arrow[r, "\pi"] & \mathcal{W}. \end{tikzcd}$$

If π and j are tor-independent and $\mathcal{N} \in \mathsf{D}(\mathcal{W}')$, then there is a natural isomorphism:

$$\mathsf{L}\pi^*\mathsf{R}j_*\mathcal{N} \simeq \mathsf{R}j'_*\mathsf{L}\pi'^*\mathcal{N}.$$

In particular, if $\mathcal{Q} \in \langle \mathsf{R}j_*\mathsf{D}^-_{\mathrm{pc}}(\mathcal{W}')\rangle$, then $\mathsf{L}\pi^*\mathcal{Q} \in \langle \mathsf{R}j'_*\mathsf{D}^-_{\mathrm{pc}}(\mathcal{Y}')\rangle$.

Proof. Now j_*, j'_* are exact and $\pi^{-1}j_* = j'_*\pi'^{-1}$. By tor-independence of π and j:

$$\begin{aligned} \mathsf{L}\pi^*\mathsf{R}j_*\mathcal{N} &= \mathcal{O}_{\mathcal{Y}} \otimes^{\mathsf{L}}_{\pi^{-1}\mathcal{O}_{\mathcal{W}}} \pi^{-1}j_*\mathcal{N} \simeq (\mathcal{O}_{\mathcal{W}} \otimes^{\mathsf{L}}_{\pi^{-1}\mathcal{O}_{\mathcal{Y}}} \pi^{-1}j_*\mathcal{O}_{\mathcal{Y}'}) \otimes^{\mathsf{L}}_{\pi^{-1}j_*\mathcal{O}_{\mathcal{Y}'}} \pi^{-1}j_*\mathcal{N} \\ &\simeq j'_*\mathcal{O}_{\mathcal{Y}'} \otimes^{\mathsf{L}}_{\pi^{-1}j_*\mathcal{O}_{\mathcal{Y}'}} \pi^{-1}j_*\mathcal{N} \\ &\simeq j'_*\mathcal{O}_{\mathcal{Y}'} \otimes^{\mathsf{L}}_{j'_*\pi'^{-1}\mathcal{O}_{\mathcal{W}'}} j'_*\pi'^{-1}\mathcal{N} \\ &\simeq j'_*(\mathcal{O}_{\mathcal{Y}'} \otimes^{\mathsf{L}}_{\pi'^{-1}\mathcal{O}_{\mathcal{W}'}} \pi'^{-1}\mathcal{N}) = \mathsf{R}j'_*\mathsf{L}\pi'^*\mathcal{N}. \end{aligned}$$

The latter claim follows from a similar argument to that in Lemma B.1. □

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E-mail address: `jackhall@math.arizona.edu`

Department of Mathematics, University of Arizona, Tucson AZ, USA