Unipotent Supports of Finite Reductive Groups

Jay Taylor

Finite Reductive Groups

Let $G$ be a connected reductive algebraic group over $K = \mathbb{F}_p$ an algebraic closure of the finite field of prime order $p > 0$.

Example. Take $G$ to be a classical matrix group: $GL_n(K)$, $SL_n(K)$, $PGL_n(K)$, $SO_n(K)$, $Sp_{2g}(K)$ or the spin groups $Spin_{2g}(K)$.

Let $F: G \to G$ be a Frobenius endomorphism and $q = p^a$ a power of $p$ determined by $F$. The fixed point group $G^F$ is called a finite reductive group.

The morphism $\phi: GL_n(K) \to GL_n(K)$ given by $E_2(\phi_{ij}) = (a_{ij})$ is a Frobenius endomorphism and $G$ is $GL_n(q)$. Take $G$ to be $SL_n(K)$, $SO_n(K)$ or $Sp_{2g}(K)$ and $F$ the restriction of $\phi$ to $G$ then $G$ is respectively $SL_n(q)$, $SO_n(q)$ or $Sp_{2g}(q)$.

If $S \subseteq G$ is a subset then we say $S$ is $F$-stable if $F(S) = S$.

Unipotent Supports

Let $\text{Irr}(G)$ denote the set of complex irreducible characters of $G$. It is a major open problem to determine the complex character tables of these groups. Much is known when $Z(G)$ is connected but the most difficult problems occur when $Z(G)$ is disconnected.

If $H$ is a group let $\text{Cl}(H)$ denote the set of conjugacy classes of $H$. When $H$ is finite we have $|\text{Irr}(H)| = |\text{Cl}(H)|$ but in general it is impossible to create a natural bijection $\text{Irr}(H) \to \text{Cl}(H)$.

Let $W: G \to GL_n(K)$ be an embedding of $G$ for some $m \geq 1$. An element $x \in G$ is called:

- semisimple if $W(x)$ is a diagonalizable matrix,
- unipotent if all the eigenvalues of $W(x)$ are 1.

Any element of $G$ can be expressed uniquely as a commuting product $x = x_0x_1x_n$ where $x_0 \in G$ is semisimple and $x = G$ is unipotent. As a consequence the study of $\text{Cl}(G)$ can be reduced to the study of semisimple and unipotent classes.

- For every $F$-stable unipotent class $O \in \text{Cl}(G)$ we fix a corresponding class representative $u \in O$.
- The fixed points $O^F$ are a disjoint union of $G$-conjugacy classes $O_i \in \text{Cl}(G)$, for $1 \leq i \leq r$. For each $i$ we fix a representative $u_i \in O_i$.
- For any $x \in G$ denote by $\alpha_2(x)$ the finite quotient group $\mathbb{Q}_2^x / G(x)^{2}$ where $G(x)$ is the centraliser of $x$ in $G$ and $\mathbb{Q}_2^x$ is its connected component.

$\Phi_2: \text{Irr}(G) \to \{F_\text{-stable unipotent classes of } G\}$ is surjective in characteristic $\text{char}(G) = 2$.

An Example of Unipotent Supports

Take $G = GL_n(K)$ and $F = E_2$ so that $G = GL_n(q)$. Consider the subgroup $B \subseteq G$ of all upper triangular matrices

$$B = \left\{ \begin{pmatrix} 1 & \cdots & \cdots & \cdots \\ \vdots & 1 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots \\ \vdots & \vdots & \vdots & 1 \end{pmatrix} \right\}. $$

This is an $F$-stable subgroup of $G$. Let $1_B \in \text{Irr}(B)$ denote the trivial character of $B = M$. The induced character decomposes as

$$\text{Ind}_{B^F}^{G^F}(1_B) = \sum_{\chi \in \text{Irr}(Z)} q^{\chi(1)} \chi \chi_\text{Ind}^G_B,$$

where $\chi_\text{Ind}$ are irreducible characters of $G$ indexed by characters of the symmetric group $\mathbb{S}_n$.

- The set of characters $\{ \chi_\varphi \mid \varphi \in \text{Irr}(\mathbb{S}_n) \}$ are the unipotent characters of $G$.

Conjugacy classes of unipotent matrices in $G$ are parameterised by partitions $\lambda \vdash n$, where the entries of $\lambda$ give the sizes of the Jordan blocks of a matrix. Let $O_\lambda$ be the class corresponding to $\lambda$. All these classes are $F$-stable.

- The elements of $\text{Irr}(O_\lambda)$ are also indexed by partitions of $n$. Let $\chi_{\lambda \mu} \in \text{Irr}(O_\lambda)$ be the character corresponding to $\mu \vdash n$.

The average value of $\chi_{\lambda}$ on $O_\lambda$ is such that

$$\text{AV}(O_\lambda, \chi_{\lambda}) = 0 \iff \lambda \in \text{dom.}$$

In particular the unipotent support of $\chi_{\lambda}$ is $O_\lambda$. This relationship comes from the Springer correspondence.

Character Degrees and Unipotent Supports

Assume now $p$ is a good prime for $G$, (it is sufficient in all cases to assume $p > 3$).

- We want to find numerical relationships between $\chi \in \text{Irr}(G)$ and $O_\lambda$.

- Characters have natural numerical invariants which arise from their degree.

Theorem (Lusztig). Let $\chi \in \text{Irr}(G)$ then there exist integers $n_0, n_1$ such that

$$\chi(1) = q^{n_1} + \text{higher powers of } q.$$

By Geck and Malle $n_1 = -\dim_{K_0}$ where $u \in O_\lambda$ and $K_0$ is the variety of all Borel subgroups containing $u$.

Lusztig made the following useful observation, (later independently verified by Lusztig and Hézard).

Theorem (Lusztig, Hézard). Assume $G(\mathbb{Z})$ is simple, $Z(G)$ is connected and $O$ is an $F$-stable unipotent class of $G$. Then $|\text{ Irr}(\chi)| = \sum_{\nu \in \text{Irr}(O)} \chi_{\nu}$, where $\chi$ runs over $O_\text{F}^{-1}(O)$.

The proof involves showing for all adjoin simple groups $G$, (which are in bijection with the Dynkin diagrams in figure 1), and each $F$-stable class $O$, (of which there are finitely many), that there exists $\chi$ such that $\chi_\lambda = 0$ and $n_0 = |\text{Irr}(\chi)|$. Again applying a case by case check we obtain the following generalisation.

Theorem (T). Assume $G$ is simple and $O$ is an $F$-stable unipotent class of $G$. Furthermore assume $\mu$ is a ‘well-chosen’ class representative then

$$|\text{Irr}(\chi)| = \sum_{\nu \in \text{Irr}(O)} \chi_{\nu},$$

where $\gamma_\nu$ is the complex character of $\gamma_\nu$.

- The multiplicity of $\chi$ in the Curtis–Alvis dual of $\gamma_\nu$ is given by $|\text{Irr}(\chi)|/|\text{Irr}(\gamma_\nu)|$.

- If $n_0 = |\text{Irr}(\chi)|$ then $\chi$ occurs with multiplicity 1 in the dual of a unique GGG of $O_\lambda$.

Again using the superset condition we obtain the following.

Theorem (T). Assume $G$ is simple, (but not a spin or half spin group), then Kawanaka’s conjecture holds if $p, q$ are large enough.

Kawanaka’s Conjecture

Let $u \in G$ be unipotent and denote by $\gamma_u$ Kawanaka’s generalised Gelfand–Graev representation (GGGR) associated to $u$. This is a representation induced from a unipotent subgroup of $G$.

- $\gamma_u$ is the regular representation of $G$.

- $\gamma_u$ is a Gelfand Graev representation whenever $u$ is a regular unipotent element.

Conjecture (Kawanaka). The set of characters of the various GGGs of $G$ form a $Z$-basis for the $\mathbb{Z}$-module of all virtual characters satisfying the condition $\chi(1) = 0$ implies $x$ is unipotent.

Using the superset condition Geck and Hézard showed Kawanaka’s conjecture holds whenever $Z(G)$ is connected and $p, q$ are large enough. Assume $O = O_\lambda$ then set

$$\gamma_\lambda := \sum_{\mu \in \text{Irr}(O)} |\text{Irr}(\chi)|/|\text{Irr}(\gamma_\nu)|.$$

where $\gamma_\nu$ is the complex character of $\gamma_\nu$.

- The multiplicity of $\chi$ in the Curtis–Alvis dual of $\gamma_\nu$ is given by $|\text{Irr}(\chi)|/|\text{Irr}(\gamma_\nu)|$.

- If $n_0 = |\text{Irr}(\chi)|$ then $\chi$ occurs with multiplicity 1 in the dual of a unique GGG associated to $O_\lambda$.

Again using the superset condition we obtain the following.