Abstract. Recently, a strong exponential character bound has been established in [4] for all elements $g \in G^F$ of a finite reductive group $G^F$ which satisfy the condition that the centraliser $C_G(g)$ is contained in a $(G,F)$-split Levi subgroup $M$ of $G$ and that $G$ is defined over a field of good characteristic. In this paper we considerably generalize this result by removing the condition that $M$ is split when the centre $Z(G)$ is connected.

1. Introduction

1.1. Assume $G$ is a connected reductive algebraic group, defined over an algebraic closure $\mathbf{F} = \mathbf{F}_p$ of the finite field $\mathbf{F}_p$ of prime order $p$, and let $F: G \to G$ be a Frobenius endomorphism corresponding to an $\mathbf{F}_q$-rational structure on $G$. The purpose of this article is to contribute to the problem of bounding the character ratios $|\chi(g)/\chi(1)|$, where $\chi \in \text{Irr}(G^F)$ is an irreducible character of the finite fixed point group $G^F$ and $g \in G^F$.

1.2. Upper bounds for absolute values of character values and character ratios in finite groups have long been of interest, particularly because of a number of applications, including to random generation, covering numbers, mixing times of random walks, the study of word maps, representation varieties and other areas. Many of these applications are connected with the well-known formula

$$\prod_{i=1}^k \frac{|C_i|}{|G|} \sum_{\chi \in \text{Irr}(G)} \frac{\chi(c_1) \cdots \chi(c_k)\chi(g^{-1})}{\chi(1)^{k-1}}$$

expressing the number of ways of writing an element $g \in G$ as a product $x_1 x_2 \cdots x_k$ of elements $x_i \in C_i$, where $C_i = c_i^G$ are $G$-conjugacy classes of elements $c_i$, $1 \leq i \leq k$, and the sum is over the set $\text{Irr}(G)$ of all irreducible characters of $G$ (see [2, 10.1]).

1.3. We are particularly interested in so-called exponential character bounds, namely bounds of the form

$$|\chi(g)| \leq \chi(1)^{\alpha_g},$$

sometimes with a multiplicative constant, holding for all characters $\chi \in \text{Irr}(G)$, where $0 \leq \alpha_g \leq 1$ depends on the group element $g \in G$. 

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1.5. Let us denote by \( U(G) \subseteq G \) the variety of unipotent elements. Following [4] we define for any \( F \)-stable Levi subgroup \( M \leq G \) a constant \( \alpha_G(M,F) \) as follows. If \( M \) is a torus then \( \alpha_G(M,F) = 0 \) otherwise we have

\[
\alpha_G(M,F) = \max_{1 \neq u \in U(M)^F} \frac{\dim u^M}{\dim u^G}
\]

where \( u^G \subseteq U(G) \) is the \( G \)-conjugacy class of \( u \), and similarly for \( M \). Note the maximum is taken over all non-identity unipotent elements.

1.6. In [4, Thm. 1.1], Bezrukavnikov, Liebeck, Shalev, and Tiep were able to obtain a bound of the form (1.4), in terms of \( \alpha_G(M,F) \), assuming that the centraliser \( C_G(g)^F \) was contained in \( M \) and \( M \) was a proper \((G,F)\)-split Levi subgroup, i.e., \( M \) is the Levi complement of an \( F \)-stable parabolic subgroup of \( G \). For the elements that the bound holds, the bound has lead to a number of interesting applications, see [4, \S 5], as well as the sequel [20].

The main result of this paper is the following, which generalizes the result of [4] to the case of a non-split Levi subgroup \( M \leq G \) assuming that \( Z(G) \) is connected.

**Theorem 1.7.** There exists a function \( f : \mathbb{N} \to \mathbb{N} \) such that the following statement holds. Assume that \( G \) is a connected reductive algebraic group of semisimple rank \( r \) with connected centre, \( F : G \to G \) is a Frobenius endomorphism, and \( p \) is a good prime for \( G \). Then for any \( F \)-stable Levi subgroup \( M \leq G \) and any element \( g \in M^F \) with \( C_G^o(g) \leq M^F \) we have that

\[
|\chi(g)| \leq f(r) \cdot \chi(1)^{\alpha_G(M,F)}
\]

for any irreducible character \( \chi \in \text{Irr}(G^F) \).

1.8. Let us consider Theorem 1.7 in the case that \( M \leq G \) is a \((G,F)\)-split Levi subgroup. In [4] it is shown that if \( g \in G^F \) is an element such that \( C_G(g)^F \subseteq M^F \) then the conclusion of Theorem 1.7 holds. However, for such an element we have \( g \in M^F \) and \( C_G^o(g) \leq M \) because \( C_G^o(g)^F \subseteq M^F \) and \( M \) is \((G,F)\)-split, see Lemma 12.5. Hence, our theorem is a generalization of the result in [4]. We also note that [4, Thm. 1.1] assumes that the derived subgroup of \( G \) is simple (but not that \( Z(G) \) is connected). However, the exact same argument used in (ii) of the proof of [4, Thm. 1.1] shows that the assumption that \( Z(G) \) is connected may be dropped from Theorem 1.7 whenever \( M \) is \((G,F)\)-split.

1.9. It appears that removing the assumption that \( Z(G) \) is connected from Theorem 1.7 cannot easily be achieved, in general, using the standard technique of regular embeddings. In principle, our proof can be applied when \( Z(G) \) is disconnected but various statements we use relating irreducible characters and characteristic functions of character sheaves remain only conjectural in this more general setting. We hope to address the case of disconnected centre groups in a sequel to this paper. The following, which applies to any finite reductive group, constitutes what can be achieved with regular embeddings and will be helpful for various applications.
Corollary 1.10. Let $G$ be a connected reductive algebraic group of semisimple rank $r$, and let $F : G \to G$ be a Frobenius endomorphism. Assume that $p$ is a good prime for $G$. Then for any $F$-stable Levi subgroup $M \leq G$ and any element $g \in M^F$ with $C_G^F(g) \leq M$ we have that

$$|\chi(g)| \leq f(r) \cdot \chi(1)^{\alpha(c(M,F))}$$

for any irreducible character $\chi \in \text{Irr}(G^F)$, with $f$ as defined in Theorem 1.7, provided that at least one of the following holds:

(a) there is a regular embedding $G \to \tilde{G}$, with a Frobenius endomorphism $\tilde{F} : \tilde{G} \to \tilde{G}$ extending $F : G \to G$, such that either $\chi$ is $\tilde{G}^F$-invariant, or $g^{\tilde{G}^F} = g^G$,

(b) $C_G^F(g)$ is connected, or

(c) $[G,G]$ is simply connected and $g$ is semisimple.

Outline of the proof of Theorem 1.7

1.11. Our proof of Theorem 1.7 follows the approach used in [4], although each step is considerably more difficult than in the split case. Associated to the Levi subgroup $M$ we have corresponding Deligne–Lusztig induction and restriction maps $R^G_M : \text{Class}(M^F) \to \text{Class}(G^F)$ and $^*R^G_M : \text{Class}(G^F) \to \text{Class}(M^F)$, which are linear maps between the spaces of class functions taking irreducible characters to virtual characters. If $M$ is $(G,F)$-split then $R^G_M$ and $^*R^G_M$ are simply Harish-Chandra induction and restriction which take irreducible characters to characters.

1.12. It is well known that under the assumptions of Theorem 1.7 we have

$$\chi(g) = ^*R^G_M(\chi)(g),$$

see Lemma 12.4. For any irreducible character $\eta \in \text{Irr}(M^F)$ and $\chi \in \text{Irr}(G^F)$ we denote by

$$m(\eta, \chi) = \langle \eta, ^*R^G_M(\chi) \rangle_{M^F} = \langle \chi, R^G_M(\eta) \rangle_{G^F} \in \mathbb{Z}$$

the multiplicity of $\eta$ in $^*R^G_M(\chi)$. Expanding $^*R^G_M(\chi)$ in terms of $\text{Irr}(M^F)$ we get from (1.13) that

$$|\chi(g)| \leq \sum_{\eta \in \text{Irr}(M^F)} |m(\eta, \chi)| \cdot \eta(1),$$

(1.14)

where we use the trivial bound $|\eta(g)| \leq \eta(1)$. This bound provides our approach to proving Theorem 1.7.

1.15. Let $\text{Irr}(M^F | ^*R^G_M(\chi))$ denote the set of irreducible constituents of $^*R^G_M(\chi)$. We start by showing that there exist three integers $f_1(G), f_2(G), f_3(G) \in \mathbb{N}$ such that all the following inequalities hold:

(a) $|\text{Irr}(M^F | ^*R^G_M(\chi))| \leq f_1(G),$

(b) $m(\eta, \chi) \leq f_2(G)$ for any $\eta \in \text{Irr}(M^F | ^*R^G_M(\chi))$, 

(c) $|\text{Irr}(M^F | ^*R^G_M(\chi))| \leq f_3(G)$. 


(c) \( \eta(1) \leq f_3(G) \cdot \chi(1)^{\alpha_G(M,F)} \) for any \( \eta \in \text{Irr}(M^F \mid {}^*R^G_M(\chi)) \).

If such integers exist then from (1.14) we clearly have that

\[ |\chi(g)| \leq f(G) \cdot \chi(1)^{\alpha_G(M,F)}. \]

where \( f(G) = f_1(G) \cdot f_2(G) \cdot f_3(G) \).

**1.16.** The definition of the integers \( f_i(G) \) will make sense for any connected reductive algebraic group. By design we will choose these integers such that \( f(G) \leq f(G_{sc}) \), where \( G_{sc} \) is the simply connected cover of the derived subgroup of \( G \). We may then define \( f(r) = \max_f f(G) \) where the maximum is taken over all semisimple and simply connected groups of rank \( r \) (there are finitely many such groups, up to isomorphism, for a fixed \( r \)).

**1.17.** Finding an integer \( f_1(G) \) satisfying 1.15(a) is achieved easily using classic results of Deligne–Lusztig. In particular, if \( W_G \) is the Weyl group of \( G \) defined with respect to some (any) maximal torus of \( G \) then we may take

\[ f_1(G) = |W_G|^2, \]

see Corollary 11.3. Our arguments here are in the spirit of those used by Lusztig [21] to prove the finiteness of the number of unipotent classes. They do not require the assumption that \( Z(G) \) is connected.

**1.18.** Finding integers satisfying 1.15(b) and 1.15(c) is appreciably more difficult. To tackle both of these problems we rely on deep results of Lusztig [27] and Shoji [32] which allow us to translate questions concerning Deligne–Lusztig induction \( R^G_M \) to corresponding questions about parabolic induction of character sheaves. These results are available to us under our assumption that \( p \) is a good prime for \( G \) and \( Z(G) \) is connected.

**1.19.** Our approach to 1.15(b) is to find an integer \( f_2'(G) \in \mathbb{N} \), depending only on the root system of \( G \), such that

\[ \langle R^G_M(\eta), R^G_M(\eta) \rangle_{G^F} = \sum_{\chi \in \text{Irr}(G^F)} m(\eta, \chi)^2 \leq f_2'(G) \]

for any irreducible character \( \eta \in \text{Irr}(M^F) \). We may then take \( f_2(G) = f_2'(G)^2 \) in 1.15(b). As we assume \( p \) is a good prime for \( G \) it is known by work of Bonnafé–Michel [7] that the Mackey formula holds. Using the Mackey formula one can obtain a function \( f_2'(G) \), as above, which is recursively defined. This approach works even when \( Z(G) \) is disconnected but the approach we now outline yields an explicit bound, which is significantly better than what can be achieved by using the Mackey formula.

**1.20.** We find \( f_2'(G) \) in two steps. First, we consider two \( F \)-stable character sheaves \( A_1, A_2 \in \text{CSh}(M) \) and their corresponding characteristic functions \( \chi_{A_1} \) and \( \chi_{A_2} \), defined with respect to some fixed Weil structure. We then obtain a bound

\[ |\langle R^G_M(\chi_{A_1}), R^G_M(\chi_{A_2}) \rangle_{G^F}| \leq |W_G| \]
by expressing the inner product in terms of coset induction in relative Weyl groups, see Proposition 12.1. Writing \( \eta \) as a linear combination of characteristic functions of character sheaves on \( M \) we are then able to bound \( \langle R_M^G(\eta), R_M^G(\eta) \rangle_G \) once we can bound the number of character sheaves involved in such a decomposition.

1.21. For this step we crucially use Shoji’s result [31] which gives us sharp control over this number. It is at this stage that the following important invariants appear. Namely, if \( H \) is an algebraic group and \( x \in H \) is an element then we denote by \( A_H(x) \) the component group \( C_H(x)/C_H^0(x) \) of the centraliser of \( x \). We set

\[
B(G) = \max_{H} \max_u |A_H(u)| \quad \text{and} \quad D(G) = \max_{H} \max_u (\dim u^H)
\]

where the maxima are taken over: all semisimple and simply connected groups \( H \) of rank at most \( r \) and all unipotent elements \( u \in \mathfrak{u}(H) \). From Shoji’s result we are able to show that we may take

\[
f^t_2(G) = B(G)^4 \cdot |W_G|,
\]

see Proposition 12.2.

1.22. Now finally let us consider 1.15(c). For each irreducible character \( \chi \in \text{Irr}(G^F) \) there is a polynomial \( D_{\chi}(t) \in \mathbb{Q}[t] \) such that \( \chi(1) = D_{\chi}(q) \). Moreover, this polynomial has the form

\[
D_{\chi}(t) = \frac{1}{n_{\chi}} (t^{a_{\chi}} + \cdots + \pm t^{a_{\chi}})
\]

where the coefficients of all intermediate powers \( t^i \), with \( a_{\chi} < i < A_{\chi} \) are integers, and \( n_{\chi} > 0 \) is an integer. By work of Lusztig [28] and Geck–Malle [16] the invariants \( n_{\chi}, A_{\chi} \), and \( a_{\chi} \), occurring in the degree polynomial have geometric interpretations.

1.23. Specifically, building on work of Lusztig, Geck–Malle have shown that to each irreducible character \( \chi \) one may associate a unique \( F \)-stable unipotent class \( 0_{\chi} = 0_{\chi}^G \) of \( G \) called the unipotent support of \( \chi \) (this can be done without any assumption on \( p \) or \( Z(G) \)). If \( p \) is a good prime for \( G \) and \( u \in 0_{\chi} \) then it is known that we have

\[
A_{\chi} = \dim 0_{\chi^*}/2, \quad a_{\chi} = \dim \mathfrak{B}_u^G, \quad \text{and} \quad n_{\chi} \mid |A_G(u)|,
\]

where \( \mathfrak{B}_u^G \) is the variety of Borel subgroups of \( G \) containing \( u \) and \( \chi^* = \pm D_{G^F}(\chi) \in \text{Irr}(G^F) \) is the Alvis–Curtis dual of \( \chi \).

1.24. This geometric interpretation explains the appearance of the \( \alpha_G(M,F) \)-bound occurring in Theorem 1.7 and also the occurrence of the term \( B(G) \) above. To achieve 1.15(c) one can now try to get a relationship between the unipotent support of \( \chi \) and the unipotent support of \( \eta \) when \( \langle \eta, R_M^G(\chi) \rangle_M^F \) is non-zero. To describe such a relationship let us write \( X \leq Y \) if \( X, Y \leq \mathfrak{u}(G) \) are subsets of the unipotent variety satisfying \( X \subseteq Y \) (the Zariski closure of \( Y \)). With this we have the following, which was shown in [4] under the additional assumption that \( M \) is \( (G,F) \)-split.

**Theorem 1.25.** Assume \( p \) is a good prime for \( G \) and \( Z(G) \) is connected. Then for any \( F \)-stable Levi subgroup \( M \leq G \) and irreducible characters \( \chi \in \text{Irr}(G^F) \) and \( \eta \in \text{Irr}(M^F) \) satisfying
\( \langle \chi, R_M^G(\eta) \rangle_{C^r} \neq 0 \) we have \( O_{\eta} \leq O_{\chi} \) and \( O_{\eta^*} \leq O_{\chi^*} \).

**Remark 1.26.** If \( M \) is a \((G, F)\)-split Levi subgroup then one can drop the assumption in Theorem 1.25 that \( Z(G) \) is connected, see Theorem 8.9.

1.27. If \( \eta \in \text{Irr}(M^F) \) and \( \chi \in \text{Irr}(G^F) \) are irreducible characters as in Theorem 1.7 then after Theorem 1.25 we get the following numerical relationship between their degree polynomials

\[
A_\eta = \dim O_M^{\eta^*}/2 \leq \dim O_G^{\chi^*}/2 = A_\chi
\]

With this relationship in hand an identical argument to that used in the proof of [4, Theorem 1.1] shows that we may take

\[
f_3(G) = 3^{D(G)/2} \cdot B(G).
\]

Thus, putting things together we see that we may choose the integer \( f(G) \), as in 1.15, to be

\[
f(G) = 3^{D(G)/2} \cdot B(G)^3 \cdot |W_G|^{\frac{3}{2}}.
\]

1.28. Our proof of Theorem 1.25 is not independent of that given in [4] and crucially uses the split case to deal with the non-split case. Indeed, it follows from work of Lusztig [28] and the first author [34] that, if \( p \) is a good prime, then to each character sheaf \( A \) one may associate a well-defined unipotent class \( O_A \subseteq U(G) \) of \( G \) which is also called its unipotent support. By relating parabolic induction of character sheaves to Harish-Chandra induction, and using the results in [4], we are able to prove the following.

**Theorem 1.29.** Assume \( p \) is a good prime for \( G \) and \( Z(G) \) is connected then for any Levi subgroup \( M \leq G \) and character sheaves \( A \in \text{CSh}(G) \) and \( B \in \text{CSh}(M) \) satisfying \( A \mid \text{ind}^G_M(B) \) we have \( O_B \leq O_A \).

1.30. Here we write \( A \mid \text{ind}^G_M(B) \) to indicate that \( A \) is a summand of the parabolically induced character sheaf \( \text{ind}^G_M(B) \). Our proof of Theorem 1.29 relies on Shoji’s work which allows us to relate unipotent supports for character sheaves and irreducible characters, see Lemma 9.7. This imposes the restriction that \( Z(G) \) is connected but it should be possible to remove this restriction. We hope to address this in the sequel. After establishing Theorem 1.29 we then use Lusztig and Shoji’s results to deduce that Theorem 1.25 holds in the non-split case.

**Outline of the Paper**

1.31. We now give a brief outline of the paper. In Section 2 we consider induction from cosets in a finite group and relate this to the usual induction from subgroups. We recall character sheaves and parabolic induction of character sheaves in Sections 3 and 4. Moreover, we we recall how parabolic induction relates to Harish-Chandra induction. Following this, in Section 5, we recall Lusztig and Shoji’s results which relate parabolic induction of cuspidal character sheaves to Deligne–Lusztig induction at the level of class functions.
1.32. In Section 6 we recall the Harish-Chandra parameterization of character sheaves and how class functions can be obtained as linear combinations of characteristic functions of cuspidal character sheaves. With this in hand we are able to establish one of our main tools in Section 7. Namely, we establish variants of Howlett–Lehrer’s comparison theorem for character sheaves and their characteristic functions. In the case of unipotently supported character sheaves these have been established by Lusztig [25, 2.4(d)] and Digne–Lehrer–Michel [9, 3.3].

1.33. Assuming $M \leq G$ is $(G, F)$-split, we show in Section 8 that Theorem 1.25 holds following Bezrukavnikov–Liebeck–Shalev–Tiep [4, 2.6]. We relate the unipotent supports of characters and character sheaves in Section 9. We are then able to prove Theorems 1.25 and 1.29 in Section 10. In Section 11 we bound the number of irreducible constituents in the Lusztig restriction of an irreducible character and, finally, in Section 12 we bound the multiplicities and prove Corollary 1.10.

1.34. At the end of the paper we include two appendices. In Appendix A we introduce some basic terminology and statements for working with $G$-invariant functions on a finite $G$-set, where $G$ is a finite group. In Appendix B we recall how the simple summands of a semisimple object $A$ of an abelian category $\mathcal{A}$ can be parameterised in terms of the simple modules of the endomorphism algebra $\text{End}_\mathcal{A}(A)$. We then study the effect of a linear functor under this parameterisation.

1.35. Notation. For any category $\mathcal{A}$ we denote by $\text{Irr}(\mathcal{A})$ the isomorphism classes of simple $\mathcal{A}$-modules. The isomorphism class of an object $A \in \mathcal{A}$ will be denoted by $[A] \in \text{Irr}(\mathcal{A})$. If $K \in \mathcal{A}$ is a semisimple object then we denote by $\text{Irr}(\mathcal{A} | K)$ the set of isomorphism classes of the simple summands of $K$. If $A$ is a $k$-algebra, with $k$ a field, then we denote by $A-\text{mod}$ the category of finite dimensional left $A$-modules.

Recall that $F = \mathbb{F}_p$. We fix an algebraic closure $\overline{\mathbb{Q}}_\ell$, where $\ell \neq p$ is a prime, and an involutive automorphism $\overline{\cdot} : \overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_\ell$ which satisfies $\overline{\zeta} = \zeta^{-1}$ for any root of unity $\zeta \in \overline{\mathbb{Q}}_\ell^\times$. If $G$ is a finite group then $\iota : G \times G \to G$ denotes the usual conjugation action, defined by $\iota(g, x) = gxg^{-1}$, and $\iota_g = \iota(g, -) : G \to G$ is the corresponding inner automorphism. We then have the space of $\overline{\mathbb{Q}}_\ell$-class functions $\text{Class}(G)$. This space has a distinguished basis $\text{Irr}(G) \subseteq \text{Class}(G)$ given by the irreducible characters of $G$. If $f \in \text{Class}(G)$ is a class function then $\text{Irr}(G | f) \subseteq \text{Irr}(G)$ denotes the set of irreducible constituents of $f$.

Throughout all varieties are assumed to be over $\mathbb{F}$. Moreover, $G$ denotes a fixed connected reductive algebraic group and $F : G \to G$ is a Frobenius endomorphism. We assume $T_0 \leq G$ is a maximal torus and $(G^*, T_0^*, F)$ is a triple dual to $(G, T_0, F)$. We denote by $W_G(T_0)$ the Weyl group $N_G(T_0)/T_0$ of $G$. If the choice of torus is irrelevant, or implicit, then we simply write $W_G$ for $W_G(T_0)$.

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2. 

2.1. Let G be a finite group and let \( \phi : G \to G \) be an automorphism then we will denote by \( G:\phi \) the semidirect product \( G \rtimes \langle \phi \rangle \) with the cyclic group \( \langle \phi \rangle \leq \text{Aut}(G) \) defined such that \( \phi g \phi^{-1} = \phi(g) \) for any \( g \in G \). We denote by \( G:\phi \) the coset \( \{ g\phi \mid g \in G \} \subseteq G:\phi \). If there is potential for ambiguity then we will write the elements of \( G:\phi \) as pairs \( (g, \phi) \).

2.2. Restricting the conjugation action \( \iota \), for \( G:\phi \), to \( G \times G:\phi \) makes \( (G:\phi, \iota) \) a finite \( G \)-set and we denote by \( \text{Class}(G:\phi) \) the space \( \text{Fun}_G(G:\phi) \), see A.2. Clearly the natural inclusion map \( G:\phi \to G:\phi \) is a \( G \)-map and we obtain corresponding induction and restriction maps

\[
\text{Ind}_{G:\phi}^{G:\phi} : \text{Class}(G:\phi) \to \text{Class}(G:\phi) \quad \text{and} \quad \text{Res}_{G:\phi}^{G:\phi} : \text{Class}(G:\phi) \to \text{Class}(G:\phi)
\]
as in Appendix A.

2.3. Let us denote by \( \text{Irr}(G:\phi \downarrow \downarrow G) \subseteq \text{Irr}(G:\phi) \) those irreducible characters whose restriction to \( G \) is irreducible. Then we define the irreducible characters of the coset \( G:\phi \) to be the elements of the set

\[
\text{Irr}(G:\phi) := \{ \text{Res}_{G:\phi}^{G:\phi}(\tilde{\eta}) \mid \tilde{\eta} \in \text{Irr}(G:\phi \downarrow \downarrow G) \} \subseteq \text{Class}(G:\phi).
\]

The set \( \text{Irr}(G:\phi) \) contains an orthonormal basis of \( \text{Class}(G:\phi) \). Such a set is obtained by choosing for each \( \phi \)-invariant character \( \eta \in \text{Irr}(G) \) an extension \( \tilde{\eta} \in \text{Irr}(G:\phi) \) and taking its restriction \( \text{Res}_{G:\phi}^{G:\phi}(\tilde{\eta}) \).

**Remark 2.4.** We will identify any irreducible character \( \tilde{\eta} \in \text{Irr}(G:\phi \downarrow \downarrow G) \) with its restriction \( \text{Res}_{G:\phi}^{G:\phi}(\tilde{\eta}) \) to avoid cumbersome notation.

2.5. Assume \( H \leq G \) is a subgroup and \( g \in G \) is an element such that \( \iota_g \phi(H) = H \) then we have \( \iota_g \phi \in \text{Aut}(H) \). We will denote by \( H:G:\phi \) the group \( H: \iota_g \phi \) and by \( H:G:\phi \) the coset \( H: \iota_g \phi \). Note that we have a surjective homomorphism \( \gamma_g : G:G:\phi \to G:\phi \), defined by \( (x, \iota_g \phi)^1 \mapsto (x_g \phi(g) \cdots \phi^{i-1}(g), \phi^1) \) which restricts to a bijective \( G \)-map \( G:G:\phi \to G:\phi \). In particular, the restriction map \( \gamma_{g}^{*} : \text{Class}(G:\phi) \to \text{Class}(G:G:\phi) \) is an isometry and we have

\[
\gamma_{g}^{*}(\text{Irr}(G:\phi \downarrow \downarrow G)) \subseteq \text{Irr}(G:G:\phi \downarrow \downarrow G) \quad \text{and} \quad \gamma_{g}^{*}(\text{Irr}(G:\phi)) \subseteq \text{Irr}(G:G:\phi).
\]

This is because \( \gamma_{g}^{*} \) is nothing other than inflation through \( \gamma_{g} \).
2.6. The restriction of $\gamma_g$ defines an injective $H$-map $\gamma_g : H.g\phi \to G.\phi$, so we obtain corresponding induction and restriction maps

$$\text{Ind}_{H.g\phi}^{G.\phi} : \text{Class}(H.g\phi) \to \text{Class}(G.\phi) \quad \text{and} \quad \text{Res}_{H.g\phi}^{G.\phi} : \text{Class}(G.\phi) \to \text{Class}(H.g\phi).$$

We will need the following lemmas concerning induction.

**Lemma 2.7.** Assume $\tilde{\chi} \in \text{Irr}(H; g\phi \downarrow H)$ and $\tilde{\rho} \in \text{Irr}(G; \phi \downarrow G)$ are irreducible characters with irreducible restrictions $\chi = \text{Res}_{H}^{H.g\phi}(\tilde{\chi}) \in \text{Irr}(H)$ and $\rho = \text{Res}_{G}^{G.\phi}(\tilde{\rho}) \in \text{Irr}(G)$ then

$$|\langle \tilde{\rho}, \text{Ind}_{H.g\phi}^{G.\phi}(\tilde{\chi}) \rangle_{G.\phi}| \leq \langle \rho, \text{Ind}_{H}^{G}(\chi) \rangle_{G}.$$ 

In particular, if $\langle \tilde{\rho}, \text{Ind}_{H.g\phi}^{G.\phi}(\tilde{\chi}) \rangle_{G.\phi} \neq 0$ then $\langle \rho, \text{Ind}_{H}^{G}(\chi) \rangle_{G} \neq 0$.

**Proof.** Let $\tilde{\psi} = \gamma_{g}^{*}(\tilde{\rho}) \in \text{Irr}(G; g\phi)$ then as $\gamma_{g}^{*}$ is an isometry and $\gamma_{g}^{*} \circ \text{Ind}_{H.g\phi}^{G.\phi} = \text{Ind}_{H.g\phi}^{H} \circ \text{Res}_{G}^{G.\phi}$, see (iii) of Proposition A.7, we have

$$\langle \tilde{\rho}, \text{Ind}_{H.g\phi}^{G.\phi}(\tilde{\chi}) \rangle_{G.\phi} = \langle \tilde{\psi}, \text{Ind}_{H.g\phi}^{G.\phi}(\tilde{\chi}) \rangle_{G.\phi}.$$ 

As we identify $\tilde{\psi}$ with its restriction $\text{Res}_{G}^{G.\phi}(\tilde{\psi})$ we have from Frobenius reciprocity and transitivity of induction, see Proposition A.7, that

$$\langle \tilde{\psi}, \text{Ind}_{H.g\phi}^{G.\phi}(\tilde{\chi}) \rangle_{G.\phi} = \langle \tilde{\psi}, \text{Ind}_{H.g\phi}^{G.\phi}(\text{Ind}_{H}^{H.g\phi}(\chi)) \rangle_{G.\phi}.$$ 

Applying [6, 1.3] to $\text{Ind}_{H}^{H.g\phi}$ we get that

$$\langle \tilde{\rho}, \text{Ind}_{H.g\phi}^{G.\phi}(\tilde{\chi}) \rangle_{G.\phi} = \sum_{\lambda \in \text{Irr}(H; g\phi \downarrow H)} \lambda(g\phi) \langle \tilde{\psi}, \text{Ind}_{H.g\phi}^{G.\phi}(\lambda \otimes \tilde{\chi}) \rangle_{G.\phi}.$$ 

As $\rho = \text{Res}_{G}^{G.\phi}(\tilde{\psi})$ an identical argument yields that

$$\langle \rho, \text{Ind}_{H}^{G}(\chi) \rangle_{G} = \langle \tilde{\psi}, \text{Ind}_{H}^{G.\phi}(\chi) \rangle_{G} = \langle \tilde{\psi}, \text{Ind}_{H.g\phi}^{G.\phi}(\text{Ind}_{H}^{H.g\phi}(\chi)) \rangle_{G}.$$ 

Moreover, a standard consequence of Clifford’s Theorem applied to $\text{Ind}_{H}^{H.g\phi}$ gives us

$$\langle \rho, \text{Ind}_{H}^{G}(\chi) \rangle_{G} = \sum_{\lambda \in \text{Irr}(H; g\phi \downarrow H)} \lambda(1) \langle \tilde{\psi}, \text{Ind}_{H.g\phi}^{G.\phi}(\lambda \otimes \tilde{\chi}) \rangle_{G.\phi},$$ 

see [18, Corollary 6.17]. Putting things together we get the desired statement as $|\lambda(g\phi)| = \lambda(1) = 1$ because $H; g\phi \downarrow H$ is cyclic.

**Corollary 2.8.** Assume $H_i \leq G$ is a subgroup and $g_i \in G$ is an element such that $t_{g_i} \phi \in \text{Aut}(H_i)$ $i \in \{1, 2\}$. If $\tilde{\chi}_i \in \text{Irr}(H_i; g_i \phi \downarrow H_i)$ is an irreducible character then

$$|\langle \text{Ind}_{H_i.g_i\phi}^{G.\phi}(\tilde{\chi}_1), \text{Ind}_{H_2.g_2\phi}^{G.\phi}(\tilde{\chi}_2) \rangle_{G.\phi}| \leq |G|.$$
Proof. Decomposing in an orthonormal basis of \( \text{Class}(G, \phi) \) we have
\[
\langle \text{Ind}^G_{H_1, g_1 \phi} (\widetilde{\chi}_1), \text{Ind}^G_{H_2, g_2 \phi} (\widetilde{\chi}_2) \rangle = \sum_{\eta \in \text{Irr}(G, \phi)} \langle \widetilde{\eta}, \text{Ind}^G_{H_1, g_1 \phi} (\chi_1) \rangle \cdot \langle \widetilde{\eta}, \text{Ind}^G_{H_2, g_2 \phi} (\chi_2) \rangle
\]
where \( \widetilde{\eta} \in \text{Irr}(G : \phi) \) is a fixed extension of \( \eta \). If \( \chi_i = \text{Res}^G_{H_i, g_i \phi} (\chi_i) \) is the irreducible restriction of \( \chi_i \) then it follows from Lemma 2.7 that
\[
|\langle \text{Ind}^G_{H_1, g_1 \phi} (\chi_1), \text{Ind}^G_{H_2, g_2 \phi} (\chi_2) \rangle| \leq \sum_{\eta \in \text{Irr}(G, \phi)} \langle \eta, \text{Ind}^G_{H_1} (\chi_1) \rangle \cdot \langle \eta, \text{Ind}^G_{H_2} (\chi_2) \rangle
\]
\[
\leq \langle \text{Ind}^G_{H_1} (\chi_1), \text{Ind}^G_{H_2} (\chi_2) \rangle.
\]
The statement now follows from the fact that \( \text{Ind}^G_{H_1} (\chi_1) \) is necessarily a summand of the character of the regular representation of \( G \).

3. Character Sheaves

3.1. Assume \( X \) is a variety equipped with an algebraic action of a connected algebraic group \( H \). We will denote by \( \mathcal{D}_H(X) \) the \( H \)-equivariant bounded derived category of \( \mathcal{O}_X \)-constructible sheaves on \( X \), as defined in [3]. Let \( X' \) be another variety equipped with an algebraic action of a connected algebraic group \( H' \). If \( \phi : X \to X' \) is an equivariant morphism then we obtain (derived) functors \( \phi^*, \phi^! : \mathcal{D}_H(X') \to \mathcal{D}_H(X) \) and \( \phi_*, \phi_! : \mathcal{D}_H(X) \to \mathcal{D}_H(X') \). We will denote by \( \mathcal{M}_H(X) \subseteq \mathcal{D}_H(X) \) the full subcategory of \( H \)-equivariant perverse sheaves on \( X \).

3.2. If \( X = H \) in 3.1 then we will implicitly assume that \( H \) acts on \( X \) by conjugation. Recall the pair \((G, F)\) fixed in 1.35. In [24, 2.10] Lusztig has defined the notion of a character sheaf which is a simple object in the category \( \mathcal{M}_G(G) \). We will denote by \( \text{CSh}(G) \subseteq \mathcal{M}_G(G) \) the full subcategory whose objects are all finite direct sums of character sheaves. We reserve the term character sheaf for a simple object of \( \text{CSh}(G) \).

3.3. We will say that a complex \( A \in \mathcal{D}_G(G) \) is \( F \)-stable if their exists an isomorphism \( \phi : F^* A \to A \). For such an \( F \)-stable complex \( A \in \mathcal{D}_G(G) \) and isomorphism \( \phi \) we denote by \( \chi_{A, \phi} \in \text{Class}(G^F) \) the characteristic function of the complex. In other words, for any \( g \in G^F \) we have
\[
\chi_{A, \phi}(g) = \sum_{i \in \mathbb{Z}} (-1)^i \text{Tr}(\phi_i, \mathcal{H}^i_g(A)),
\]
where \( \mathcal{H}^i_g(A) \) denotes the stalk at \( g \) of the \( i \)th cohomology sheaf of \( A \). The map \( A \mapsto F^* A \) defines a permutation of the isomorphism classes \( \text{Irr}(\mathcal{M}_G(G)) \) and \( \text{Irr}(\text{CSh}(G)) \) and we denote by \( \text{Irr}(\mathcal{M}_G(G))^F \subseteq \text{Irr}(\mathcal{M}_G(G)) \) and \( \text{Irr}(\text{CSh}(G))^F \subseteq \text{Irr}(\text{CSh}(G)) \) the corresponding sets of fixed points. In other words, we have \( \text{Irr}(\text{CSh}(G))^F \) are the isomorphism classes of \( F \)-stable character sheaves. The following result of Lusztig is shown under some mild restrictions in [24, §25] and is established in full generality in [29].

Theorem 3.4 (Lusztig). There exists a family of isomorphisms \((\phi_A : F^* A \to A)_{[A] \in \text{Irr}(\text{CSh}(G))^F}\) such that the set of characteristic functions \( \{\chi_{A, \phi_A} \mid [A] \in \text{Irr}(\text{CSh}(G))^F\} \) forms an orthonormal
basis of \( \text{Class}(G^F) \). Each \( \phi_A \) is defined uniquely up to multiplication by a root of unity.

**Remark 3.5.** If \( A \in \mathcal{D}_G(G) \) is an \( F \)-stable complex then we will often write \( X_A \) instead of \( X_{A,\phi} \) with an isomorphism \( \phi : F^*A \to A \) implicitly chosen. If \( A \in \text{CSh}(G) \) is a character sheaf then we will always assume that \( \phi \) is chosen to be part of such a family as in Theorem 3.4.

4. **Parabolic Induction**

4.1. Let \( P \leq G \) be a parabolic subgroup with unipotent radical \( U \leq P \) and Levi complement \( L \leq P \). Associated to this data we have a parabolic induction functor \( \text{ind}_L^G : \mathcal{D}_L(L) \to \mathcal{D}_G(G) \) defined as follows, see [24, §4.1]. First, consider the diagram

\[
L \xleftarrow{\pi} \hat{X} \xrightarrow{\sigma} \hat{X} \xrightarrow{\tau} G
\]

where we have

\[
\hat{X} = \{(g, h) \in G \times G \mid h^{-1}gh \in P\} \\
\hat{X} = \{(g, hP) \in G \times (G/P) \mid h^{-1}gh \in P\} \\
\pi(g, h) = \pi_P(h^{-1}gh) \\
\sigma(g, h) = (g, hP) \\
\tau(g, hP) = g
\]

and \( \pi_P : P \to L \) is the canonical projection map. Here \( \hat{X} \) is a variety where \( G \) acts on the left via \( x \cdot (g, h) = (xgx^{-1}, xh) \) and \( P \) acts on the right by \( (g, h) \cdot y = (g, hy) \). Hence, we have an action of \( G \times P^{op} \) on \( \hat{X} \) where \( P^{op} \) is the opposite group of \( P \). Moreover, \( \hat{X} \) is the quotient of \( \hat{X} \) by the right \( P \)-action. All the morphisms are equivariant with respect to the stated actions.

4.3. The fibres of \( \pi \) have dimension \( \dim G + \dim U \) and we set \( \bar{\pi} := \pi^*[\dim G + \dim U] \). Similarly, the fibres of \( \sigma \) have dimension \( \dim P \) and we set \( \bar{\sigma} := \sigma^*[\dim P] \). If \( A \in \mathcal{D}_L(L) \) then there exists a canonical complex \( D \in \mathcal{D}_{G \times P^{op}}(\hat{X}) \) such that \( \bar{\pi}A = \bar{\delta}D \). We then define \( \text{ind}_L^G(A) := \bar{\tau}D \). If \( f \in \text{Hom}_{\mathcal{D}_L(L)}(A, B) \) is a morphism then we get a morphism \( \text{ind}_L^G(f) \in \text{Hom}_{\mathcal{D}_G(G)}(\text{ind}_L^G(A), \text{ind}_L^G(B)) \) as follows. We have an induced morphism \( \bar{\pi}f : \bar{\pi}A \to \bar{\pi}B \). As \( \sigma \) is smooth with connected fibres we have \( \bar{\sigma} \) is a fully faithful functor so there exists a unique morphism \( \bar{f}' \) such that \( \bar{\pi}f = \bar{\sigma}\bar{f}' \). We then have \( \text{ind}_L^G(f) = \bar{\tau}f' \). We will need the following concerning induction, which is noted in the proof of [24, 15.2].

**Lemma 4.4.** Assume \( P, L, \) and \( A \) are \( F \)-stable. Let \( \phi : F^*A \to A \) be an isomorphism and let \( \hat{\phi} = \text{ind}_L^G(\phi) \) be the induced isomorphism \( F^* \text{ind}_L^G(A) \to \text{ind}_L^G(\phi) \) then

\[
\mathcal{R}_{L}^G(X_{A,\phi}) = X_{\text{ind}_L^G(A),\hat{\phi}}.
\]

**Proof.** As \( L \) and \( P \) are \( F \)-stable we have \( F^* \text{ind}_L^G(A) = \text{ind}_L^G(A) \), see [33, 3.9(a)], so we can view \( \hat{\phi} \) as a morphism \( F^* \text{ind}_L^G(A) \to \text{ind}_L^G(\phi) \). Let \( \phi' \) be the morphism satisfying \( \bar{\pi}\phi = \bar{\delta}\phi' \) then by the function-sheaf dictionary, see [19, Theorem 12.1], we
have for any \( g \in G \) that

\[
X_{\text{ind}_E(A),\phi}(g) = \sum_{x \in X^F, \tau(x) = g} X_{D,\phi}^*(x) = \sum_{hP \in (G/P)^F, h^{-1}gh \in P} X_{D,\phi}^*(g, hP),
\]

\[
X_{\pi A, \pi \phi}(g, h) = (-1)^{\dim G + \dim U} X_{A, \phi}(\pi P(h^{-1}gh)),
\]

\[
X_{\sigma D, \sigma \phi}(g, h) = (-1)^{\dim P} X_{D, \phi}^*(g, hP).
\]

Note that \( \dim G + \dim U - \dim P = |\Phi| - |\Phi_L| \) where \( \Phi \), resp., \( \Phi_L \), are the roots of \( G \), resp., \( L \), with respect to some maximal torus. Hence, we must have \( \dim G + \dim U \equiv \dim P \mod 2 \) so \( X_{D,\phi}^*(g, hP) = X_{A, \phi}(\pi P(h^{-1}gh)) \). As \( P \leq G \) is closed and connected the natural map \( G^F/P^F \to (G/P)^F \) is a bijection. Combining these identities gives

\[
X_{\text{ind}_E(A),\phi}(g) = \sum_{hP \in G^F/P^F, h^{-1}gh \in P} X_{A, \phi}(\pi P(h^{-1}gh)) = \mathbb{R}_G^L(X_{A, \phi})(g).
\]

### 4.5. We assume now that \( K \in \mathcal{M}_G(G) \) is a semisimple object with endomorphism algebra \( A = \text{End}_{\mathcal{M}_G(G)}(K) \). We then have a functor \( \mathfrak{F}_K = \text{Hom}_{\mathcal{M}_G(G)}(-, K) : \mathcal{M}_G(G) \to A\text{-mod} \) as in Appendix B. Assume \( K \) is F-stable, i.e., there exists an isomorphism \( \phi : F^*K \to K \), then we have an algebra automorphism \( \sigma : A \to A \) given by \( \sigma(\theta) = \phi \circ F^*\theta \circ \phi^{-1} \). For any \( A \)-module \( E \in A\text{-mod} \) we denote by \( E_{\sigma} \) the module equal to \( E \) as a vector space but with the action defined by \( a \cdot e = \sigma^{-1}(a) \cdot e \). The following is a straightforward consequence of Lemma B.6.

**Lemma 4.6.** For any summand \( A \mid K \) we have an isomorphism \( \rho : \mathfrak{F}_K(A)_{\sigma} \to \mathfrak{F}_K(F^*A) \) of \( A \)-modules defined by \( \rho(f) = \phi \circ F^*(f) \). In particular, the assignment \( \phi \mapsto \rho^{-1} \circ \mathfrak{F}_K(\phi) \) defines a bijection between the isomorphisms \( F^*A \to A \) in \( \mathcal{M}_G(G) \) and the \( A \)-module isomorphisms \( \mathfrak{F}_K(A) \to \mathfrak{F}_K(A)_{\sigma} \).

### 4.7. Note that as \( K \) is F-stable we have the assignment \( A \mapsto F^*A \) defines a permutation of \( \text{Irr}(\mathcal{M}_G(G) \mid K) \), c.f., 1.35, and we denote by \( \text{Irr}(\mathcal{M}_G(G) \mid K)^F \subseteq \text{Irr}(\mathcal{M}_G(G) \mid K) \) the set of fixed points under this permutation. If \( \phi_A : F^*A \to A \) is an isomorphism in \( \mathcal{M}_G(G) \) then we have a corresponding isomorphism denoted by \( \sigma_A = \mathfrak{F}_K(\phi_A^{-1}) \circ \rho : \mathfrak{F}_K(A)_{\sigma} \to \mathfrak{F}_K(A) \). By definition we have \( \sigma_A(f) = \phi \circ F^*(f) \circ \phi_A^{-1} \) for any \( f \in \mathfrak{F}_K(A) \). An identical argument to that used in [24, 10.4.2] yields the following.

**Lemma 4.8.** Assume \( K \in \mathcal{M}_G(G) \) is an F-stable semisimple perverse sheaf, as above, then we have

\[
X_{K, \phi} = \sum_{A \in \text{Irr}(\mathcal{M}_G(G) \mid K)^F} \text{Tr}(\sigma_A, \mathfrak{F}_K(A)) X_{A, \phi_A}.
\]

Moreover, we have the trace \( \text{Tr}(\sigma_A, \mathfrak{F}_K(A)) \) is non-zero for any \( A \in \text{Irr}(\mathcal{M}_G(G) \mid K)^F \) as \( \sigma_A \) is an automorphism of the vector space \( \mathfrak{F}_K(A) \).
5. Inducing Cuspidal Character Sheaves

5.1. We denote by Cusp($G$) the set of triples $(L, \Sigma, [\mathcal{E}])$ where: $L \subseteq G$ is a Levi subgroup, $\Sigma \subseteq L$ is the inverse image of an isolated conjugacy class under the natural projection map $L \to L/Z^\Sigma(L)$, and $[\mathcal{E}]$ is the isomorphism class of an irreducible $L$-equivariant local system $\mathcal{E}$ on $\Sigma$ such that
\[
\mathcal{E}^\dagger := IC(\Sigma, \mathcal{E})[\dim \Sigma] \in \text{CSh}(L)
\]
is a cuspidal character sheaf. For brevity we will write $(L, \Sigma, \mathcal{E})$ for the tuple $(L, \Sigma, [\mathcal{E}])$ with it implicitly assumed that $\mathcal{E}$ is taken up to isomorphism. Here we use the notation of Lusztig [27, 1.4], except we have shifted the intersection cohomology complex to make it perverse.

Remark 5.2. We note that if $M \subseteq G$ is a Levi subgroup of $G$ then we have a natural inclusion Cusp($M$) $\subseteq$ Cusp($G$) of cuspidal triples.

5.3. To each tuple $(L, \Sigma, \mathcal{E}) \in$ Cusp($G$) we associate a perverse sheaf $\mathcal{K}_{L,\Sigma,\mathcal{E}}^G \in \mathcal{M}_G(G)$ as follows. Let $\Sigma_{\text{reg}} = \{g \in \Sigma \mid C_G^G(g_{\text{ss}}) \subseteq L\}$, where $g_{\text{ss}}$ denotes the semisimple part of $g$, and set $Y = \bigcup_{g \in G} g\Sigma_{\text{reg}} g^{-1}$ then we have a diagram
\[
\begin{array}{c}
L & \xleftarrow{\alpha} & \hat{Y} & \xrightarrow{\beta} & \check{Y} & \xrightarrow{\gamma} & Y \\
\end{array}
\] (5.4)

where
\[
\hat{Y} = \{(g, h) \in G \times G \mid h^{-1} gh \in \Sigma\}, \quad \check{Y} = \{(g, hL) \in G \times (G/L) \mid h^{-1} gh \in L\},
\]
\[
\alpha(g, h) = h^{-1} gh, \quad \beta(g, hL) = (g, hL), \quad \gamma(g, hL) = g.
\]

As for parabolic induction we have $\hat{Y}$ is a variety where $G$ acts on the left via $x \cdot (g, h) = (xgx^{-1}, xh)$ and $L$ acts on the right via $(g, hL) \cdot l = (gl, hL l)$. We have $\check{Y}$ is the quotient of $Y$ by the right $L$-action. Now, there exists a unique $G$-equivariant local system $\mathcal{E}$ on $\hat{Y}$ such that $\alpha^* \mathcal{E} = \beta^* \mathcal{E}$. The $G$-equivariant local system $\gamma^* \mathcal{E}$ is semisimple, see [23, Proposition 3.5], and we set $\mathcal{K}_{L,\Sigma,\mathcal{E}}^G = \text{IC}([\check{Y}, \gamma^* \mathcal{E})[\dim Y]$ viewed as a perverse sheaf on $G$ via extension by 0.

Theorem 5.5 (Lusztig, [24, II, 4.3.2, 8.2.3]). The perverse sheaves $\mathcal{K}_{L,\Sigma,\mathcal{E}}^G$ and $\text{ind}^G_{\mathcal{L} \subseteq \mathcal{P}}(\mathcal{E}^\dagger)$ are semisimple and canonically isomorphic. Moreover, all their simple summands are character sheaves.

5.6. We have a right action of $G$ on Cusp($G$) defined by
\[
(L, \Sigma, \mathcal{E}) \cdot g = (1_g^{-1}(L), 1_g^{-1}(\Sigma), 1^*_g \mathcal{E}).
\]
The orbit of $(L, \Sigma, \mathcal{E})$ under this action will be denoted by $[L, \Sigma, \mathcal{E}]$ and the set of all orbits will be denoted by Cusp($G$). By [24, 7.1.12, 7.6] and Theorem 5.5 we have a
decomposition
\[ \text{Irr}(\text{CSh}(G)) = \bigsqcup_{[L, \Sigma, \mathcal{E}] \in \text{Cusp}(G)} \text{Irr}(\text{CSh}(G) \mid K_{L, \Sigma, \mathcal{E}}^G). \]

5.7. The Frobenius endomorphism $F$ defines a permutation of the set $\text{Cusp}(G)$ via the map $(L, \Sigma, \mathcal{E}) \mapsto (F^{-1}(L), F^{-1}(\Sigma), F^* \mathcal{E})$. We denote by $\text{Cusp}(G)^F$ the set of fixed points. Now assume $(L, \Sigma, \mathcal{E}) \in \text{Cusp}(G)^F$ is $F$-fixed then, by definition, there exists an isomorphism $\varphi : F^* \mathcal{E} \to \mathcal{E}$ (recall that $\mathcal{E}$ is taken up to isomorphism). We will denote by $\text{Cusp}(G, F)$ the set of tuples $(L, \Sigma, \mathcal{E}, \varphi)$ with $(L, \Sigma, \mathcal{E}) \in \text{Cusp}(G)^F$ and $\varphi : F^* \mathcal{E} \to \mathcal{E}$ an isomorphism; such tuples are called induction data in [27, 1.8]. The isomorphism $\varphi$ naturally extends to an isomorphism $\varphi^\#: F^* \mathcal{E}^\# \to \mathcal{E}^\#$ by the functoriality of intersection cohomology which, in turn, extends to an isomorphism $\hat{\varphi}^\#: F^* K_{L, \Sigma, \mathcal{E}}^G \to K_{L, \Sigma, \mathcal{E}}^G$, see [24, 8.2]. We will need the following powerful generalization of Lemma 4.4 to the case of non-split Levi subgroups.

**Theorem 5.8 (Lusztig [27, Proposition 9.2], Shoji [32, Theorem 4.2]).** Assume $p$ is a good prime for $G$ and $Z(G)$ is connected then for any tuple $(L, \Sigma, \mathcal{E}, \varphi) \in \text{Cusp}(G, F)$ we have

\[ R^G_L(X_{\mathcal{E}^\#, \varphi^\#}) = X_{K_{L, \Sigma, \mathcal{E}}^G, \hat{\varphi}^\#}. \]

5.9. The permutation $\text{Cusp}(G) \to \text{Cusp}(G)$ induced by $F$ also induces a permutation of the $G$-orbits $\text{Cusp}(G)$ and we again denote by $\text{Cusp}(G)^F$ the set of fixed points. A standard argument using the Lang–Steinberg theorem shows that the canonical map $\text{Cusp}(G)^F \to \text{Cusp}(G)^F$ is surjective, see [24, 10.5]. Moreover, we have a decomposition

\[ \text{Irr}(\text{CSh}(G))^F = \bigsqcup_{[L, \Sigma, \mathcal{E}] \in \text{Cusp}(G)^F} \text{Irr}(\text{CSh}(G) \mid K_{L, \Sigma, \mathcal{E}}^G)^F. \]

If $\text{Class}(G^F \mid [L, \Sigma, \mathcal{E}]) \subseteq \text{Class}(G^F)$ denotes the subspace spanned by the characteristic functions of the character sheaves contained in $\text{Irr}(\text{CSh}(G) \mid K_{L, \Sigma, \mathcal{E}}^G)^F$ then we have a corresponding direct sum decomposition

\[ \text{Class}(G^F) = \bigoplus_{[L, \Sigma, \mathcal{E}] \in \text{Cusp}(G)^F} \text{Class}(G^F \mid [L, \Sigma, \mathcal{E}]). \]

6. **Harish-Chandra Parameterization of Character Sheaves**

6.1. We will assume fixed a triple $(L, \Sigma, \mathcal{E}) \in \text{Cusp}(G)$ and a parabolic subgroup $P \leq G$ with Levi complement $L$. We denote by $N_G(L, \Sigma, \mathcal{E}) \subseteq G$ the stabiliser of the triple under the $G$-action described in 5.1. We clearly have $L \leq N_G(L, \Sigma, \mathcal{E}) \leq N_G(L)$ and so we obtain a subgroup $W_G(L, \Sigma, \mathcal{E}) := N_G(L, \Sigma, \mathcal{E})/L$ of the relative Weyl group $W_G(L) := N_G(L)/L$.

6.2. Let us denote by $A_{L, \Sigma, \mathcal{E}}^G$ the endomorphism algebra $\text{End}_{G(G)}(K_{L, \Sigma, \mathcal{E}}^G)$, which is a finite dimensional $\overline{Q}_\ell$-algebra. In [24, 10.2] Lusztig has shown that the algebra $A_{L, \Sigma, \mathcal{E}}^G$ is isomorphic to the group algebra $\overline{Q}_\ell[W_G(L, \Sigma, \mathcal{E})]_\alpha$ twisted by a 2-cocycle $\alpha$. We will
need to assume that this 2-cocycle is trivial. In other words the following property holds:

\((\mathcal{P}_{L,\Sigma,\mathcal{E}}^G, \mathcal{A}_{L,\Sigma,\mathcal{E}}^G)\) the algebras \(\mathcal{A}_{L,\Sigma,\mathcal{E}}^G\) and \(\mathcal{Q}_l[\mathcal{W}_G(L,\Sigma,\mathcal{E})]\) are isomorphic.

This assumption will not prove to be restrictive as \((\mathcal{P}_{L,\Sigma,\mathcal{E}}^G, \mathcal{A}_{L,\Sigma,\mathcal{E}}^G)\) is known to hold in several important cases. For instance, if \(\Sigma\) contains a unipotent element then \((\mathcal{P}_{L,\Sigma,\mathcal{E}}^G, \mathcal{A}_{L,\Sigma,\mathcal{E}}^G)\) holds by [23, Theorem 9.2]. Moreover, we have the following in the connected centre case, see [31, I, Lemma 5.9] and the remarks in the proof of [32, Theorem 4.2].

**Proposition 6.3 (Lusztig, Shoji).** Assume \(p\) is a good prime for \(G\) and \(Z(G)\) is connected then for any triple \((L,\Sigma,\mathcal{E}) \in \text{Cusp}(G)\) we have \((\mathcal{P}_{L,\Sigma,\mathcal{E}}^G, \mathcal{A}_{L,\Sigma,\mathcal{E}}^G)\) holds.

6.4. To each element \(w \in \mathcal{W}_G(L,\Sigma,\mathcal{E})\) Lusztig has defined an invertible endomorphism \(\Theta_w^G \in \mathcal{A}_{L,\Sigma,\mathcal{E}}^G\) as follows. Using the notation of 5.3 let \(\gamma_w : \tilde{Y} \to \tilde{Y}\) be defined by \(\gamma_w(g, x L) = (g, x w^{-1} L)\) where \(w \in N_G(L,\Sigma,\mathcal{E})\) is a fixed representative of \(w \in \mathcal{W}_G(L,\Sigma,\mathcal{E})\). There exists an isomorphism \(\theta_w : \mathcal{E} \to \mathcal{E}_w\) of \(L\)-equivariant local systems and this extends uniquely to an isomorphism \(\delta_w^G : \mathcal{E} \to \gamma_w^* \mathcal{E}\) satisfying \(\alpha^* \delta_w^G = \beta^* \delta_w^G\), see [23, Proposition 3.5]. As \(\gamma_* \gamma_w^* = \gamma_*\) we have \(\gamma_* \delta_w^G\) is an automorphism of \(\gamma_* \mathcal{E}\). Applying the fully faithful functor \(\text{IC}(\mathcal{Y},-)[\dim \mathcal{Y}]\) to \(\gamma_* \delta_w^G\) we obtain an invertible endomorphism \(\Theta_w^G \in \mathcal{A}_{L,\Sigma,\mathcal{E}}^G\). The following is easy, see the end of [23, 9.4].

**Lemma 6.5.** If \((\mathcal{P}_{L,\Sigma,\mathcal{E}}^G, \mathcal{A}_{L,\Sigma,\mathcal{E}}^G)\) holds then \(\mathcal{A}_{L,\Sigma,\mathcal{E}}^G\) admits a 1-dimensional representation, say \(E\), and there exists a unique family of isomorphisms \((\theta_w : \mathcal{E} \to \mathcal{E}_w)_{w \in \mathcal{W}_G(L,\Sigma,\mathcal{E})}\) such that \(\Theta_w^G\) acts trivially on \(E\). The map \(w \to \Theta_w^G\) defines an algebra isomorphism \(\mathcal{Q}_l[\mathcal{W}_G(L,\Sigma,\mathcal{E})] \to \mathcal{A}_{L,\Sigma,\mathcal{E}}^G\).

6.6. Note one also gets an invertible endomorphism \(\hat{\Theta}_w^G\) of the complex \(\mathcal{E}^\sharp\) by applying the fully faithful functor \(\text{IC}(\Sigma,-)[\dim \Sigma]\) to \(\theta_w\). Applying induction one then obtains an invertible endomorphism \(\text{ind}_{L,C}^G(\hat{\Theta}_w^G)\) of the complex \(\text{ind}_{L,C}^G(\mathcal{E}^\sharp)\). We will need the following compatibility between these two constructions.

**Lemma 6.7.** The invertible endomorphisms \(\text{ind}_{L,C}^G(\hat{\Theta}_w^G)\) and \(\Theta_w^G\) coincide under the isomorphism \(\text{ind}_{L,C}^G(\mathcal{E}^\sharp) \cong \mathcal{K}_{L,\Sigma,\mathcal{E}}^G\) of Theorem 5.5.

**Proof.** We freely use the notation of 4.1 and 5.3. Let us denote by \(\iota_Y : Y \to G\) and \(\iota : \tau^{-1}(Y) \to \tilde{X}\) the natural inclusion morphisms. The endomorphism \(\Theta_w^G\) is uniquely determined by the property that \(\iota_Y^* \Theta_w^G\) is \(\gamma_* \delta_w^G[\dim Y]\). Hence it suffices to show that the morphism corresponding to \(\text{ind}_{L,C}^G(\hat{\Theta}_w^G)\) has this property.

We denote by \(f\) the unique morphism satisfying \(\pi \hat{\Theta}_w^G = \delta f\) then, by definition, we have \(\text{ind}_{L,C}^G(\hat{\Theta}_w^G) = \tau f\). Lusztig has shown that we have an isomorphism \(\kappa : \tilde{Y} \to \tau^{-1}(Y)\) defined by \(\kappa(g, hL) = (g, hP)\), see [23, 4.3(c)]. The isomorphism between \(\text{ind}_{L,C}^G(\mathcal{E}^\sharp)\) and \(\mathcal{K}_{L,\Sigma,\mathcal{E}}^G\) gives an isomorphism

\[\iota_Y^* \text{ind}_{L}^G(\mathcal{E}^\sharp) = \iota_Y^* \tau \iota_D \cong \iota_Y^* \mathcal{K}_{L,\Sigma,\mathcal{E}}^G = \gamma_* \mathcal{E},\]

[23, 4.4, 4.5]. Now \(\iota_Y^* \tau f = \tau j^* f\). Moreover, by [23, 4.3(b)] we have \(\gamma\) is proper so by smooth base change \(\tau j = \gamma_* \kappa^*\) because \(\gamma = \tau \circ \kappa\). Under the above isomorphism \(\iota_Y^* \tau j f\) corresponds to \(\gamma_* \kappa^* f\). Hence, it suffices to show that \(\kappa^* f = \delta_w^G[\dim Y]\).
Recall from [23, 4.3(a)] that we have an equality
\[ \dim G + \dim U - \dim P = \dim Y - \dim \Sigma. \]
As \( \sigma \) = \( \pi \hat{U}_w \) we get that \( \sigma^* f = \pi^* \hat{U}_w[\dim Y - \dim \Sigma] \). If \( \iota_Y : Y \to X \) is the natural inclusion map then \( \iota \circ \kappa \circ \beta = \sigma \circ \iota_Y \) which implies that
\[ \beta^* \kappa^* f = \iota_Y^* \sigma^* f = \iota_Y^* \pi^* \hat{U}_w[\dim Y - \dim \Sigma]. \]
The image of the morphism \( \pi \circ \iota_Y \) is contained in \( \Sigma \) so agrees with \( \alpha \). Hence, if \( \iota_\Sigma : \Sigma \to L \) is the natural inclusion map then we have \( \iota_\Sigma^* \pi^* \hat{U}_w \) coincides with \( \alpha^* \iota_\Sigma^* \hat{U}_w \) but by definition \( \iota_\Sigma^* \hat{U}_w = \theta_w[\dim \Sigma] \). Putting things together we get that \( \beta^* \kappa^* f = \alpha^* \theta_w[\dim Y] \) which implies that \( \kappa^* f = \hat{U}_w[\dim Y] \), as desired. □

6.8. Recall from Appendix B that we have a functor
\[ \mathcal{G}^G_{\Sigma, \varepsilon} = \text{Hom}_{\mathcal{M}(G)}(-, K^G_{\Sigma, \varepsilon}) : \mathcal{M}(G) \to A^G_{\Sigma, \varepsilon} - \text{mod}. \]
As \( (\mathcal{G}^G_{\Sigma, \varepsilon}) \) holds we may view this as a functor \( \mathcal{M}(G) \to \overline{Q}_{\ell}[W_G(L, \Sigma, \varepsilon)] - \text{mod} \). For any character \( \eta \in \text{Class}(W_G(L, \Sigma, \varepsilon)) \) we will denote by \( K^G_{\eta} \in \text{CSh}(G, [L, \Sigma, \varepsilon]) \) a perverse sheaf such that \( \mathcal{G}_{\Sigma, \varepsilon}(K^G_{\eta}) \) affords the character \( \eta \). This yields a bijection
\[ \text{Irr}(W_G(L, \Sigma, \varepsilon)) \to \text{Irr}(\text{CSh}(G) \mid K^G_{\Sigma, \varepsilon}) \]
as in Lemma B.5.

6.9. Now assume \( (L, \Sigma, \varepsilon, \varphi) \in \text{Cusp}(G, F) \). Recall that the isomorphism \( \varphi : F^* \varepsilon \to \varepsilon \) induces an isomorphism \( \phi^\sharp : F^* K^G_{\Sigma, \varepsilon} \to K^G_{\Sigma, \varepsilon} \). From this we obtain a corresponding algebra automorphism \( \sigma : A^G_{\Sigma, \varepsilon} \to A^G_{\Sigma, \varepsilon} \) as in 4.5. It is straightforward to check that \( \sigma(G^w) = \Theta_{F^{-1}(w)}^G \) for all \( w \in W_G(L, \Sigma, \varepsilon) \). It follows from Lemma 4.6 that the bijection above restricts to a bijection
\[ \text{Irr}(W_G(L, \Sigma, \varepsilon))^F \to \text{Irr}(\text{CSh}(G) \mid K^G_{\Sigma, \varepsilon})^F. \]

6.10. Assume \( \eta \in \text{Irr}(W_G(L, \Sigma, \varepsilon))^F \) is an \( F \)-stable character and let \( \Lambda = K^G_{\eta} \) then there exists an isomorphism \( \phi_{\Lambda} : F^* \Lambda \to \Lambda \). As in 4.7 we obtain a corresponding isomorphism \( \sigma_{\Lambda} : A^G_{\Sigma, \varepsilon}(\Lambda) \to \mathcal{G}_{\Sigma, \varepsilon}(\Lambda) \). We make \( \mathcal{G}^G_{\Sigma, \varepsilon}(\Lambda) \) into a \( \overline{Q}_{\ell}[W_G(L, \Sigma, \varepsilon), F] \)-module by setting \( F \cdot v = \sigma_{\Lambda}^{-1}(v) \). This module then affords an irreducible character \( \tilde{\eta} \in W_G(L, \Sigma, \varepsilon)^F \) which extends \( \eta \). We will denote by \( \phi_{\tilde{\eta}} : F^* K^G_{\tilde{\eta}} \to K^G_{\tilde{\eta}} \) an isomorphism such that \( \mathcal{G}_{\Sigma, \varepsilon}(K^G_{\tilde{\eta}}) \) affords the character \( \tilde{\eta} \) when viewed as a \( \overline{Q}_{\ell}[W_G(L, \Sigma, \varepsilon), F] \)-module.

Remark 6.11. We may, and will, assume that \( \phi_{\tilde{\eta}} \) is part of a family of isomorphisms as in Theorem 3.4.
6.12. For each element \( w \in W_G(L, \Sigma, \mathcal{E}) \) recall our choice of representative \( \hat{w} \in N_G(L, \Sigma, \mathcal{E}) \) from 6.4. In addition let us choose an element \( g_w \in G \) such that \( g_w^{-1} F(g_w) = w^{-1} \); such an element exists by the Lang–Steinberg theorem. We then obtain a new tuple \( (L_w, \Sigma_w, \mathcal{E}_w, \varphi_w) \in \text{Cusp}(G, F) \) where

\[
L_w = t_{g_w}(L) \quad \Sigma_w = t_{g_w}(\Sigma) \quad \mathcal{E} : = (t_{g_w}^{-1})^* \mathcal{E}
\]

and \( \varphi_w : F^* \mathcal{E}_w \to \mathcal{E}_w \) is an isomorphism determined by \( \varphi \), as in [24, 10.6].

6.13. Using the fact that \( (wF^{-1})^{-1} = Fw^{-1} = w(w^{-1}F)w^{-1} \), where this computation takes place inside \( W_G(L, \Sigma, \mathcal{E}) : F \), we have by [24, 10.4.5] that

\[
\mathcal{X}_{K_n, \eta \varphi} = \frac{1}{|W_G(L, \Sigma, \mathcal{E})|} \sum_{w \in W_G(L, \Sigma, \mathcal{E})} \tilde{\eta}(w^{-1}F) \mathcal{X}_{K^G_{L(L, \Sigma, \mathcal{E})_w}, \eta \varphi_w} \tag{6.14}
\]

for any \( \eta \in \text{Irr}(W_G(L, \Sigma, \mathcal{E}))^F \) and extension \( \tilde{\eta} \in \text{Irr}(W_G(L, \Sigma, \mathcal{E}) : F) \). Inspired by this we define a \( \mathbb{Q}_L \)-linear map \( \mathcal{R}_{L, \Sigma, \mathcal{E}, \varphi} : \text{Class}(W_G(L, \Sigma, \mathcal{E}), F) \to \text{Class}(G^F | [L, \Sigma, \mathcal{E}]) \) by setting

\[
\mathcal{R}_{L, \Sigma, \mathcal{E}, \varphi}(f) = \frac{1}{|W_G(L, \Sigma, \mathcal{E})|} \sum_{w \in W_G(L, \Sigma, \mathcal{E})} f(w^{-1}F) \mathcal{R}_{L_w}^G(\chi_{\mathcal{E}_w, \eta \varphi_w}).
\]

This construction has the following property.

**Lemma 6.15.** Assume \( (L, \Sigma, \mathcal{E}, \varphi) \in \text{Cusp}(G, F) \) is a fixed tuple. If \( p \) is a good prime for \( G \) and \( Z(G) \) is connected then the following two conditions hold:

(a) for any irreducible character \( \tilde{\eta} \in \text{Irr}(W_G(L, \Sigma, \mathcal{E}) : F) \) we have

\[
\mathcal{R}_{L, \Sigma, \mathcal{E}, \varphi}(\tilde{\eta}) = \mathcal{X}_{K_n, \varphi \eta},
\]

(b) \( \mathcal{R}_{L, \Sigma, \mathcal{E}, \varphi} \) is an isometry onto its image.

**Proof.** (a). This follows from (6.14) and the definition of \( \mathcal{R}_{L, \Sigma, \mathcal{E}, \varphi} \) by applying Theorem 5.8 to each tuple \( (L_w, \Sigma_w, \mathcal{E}_w, \varphi_w) \).

(b) This is clear as \( \mathcal{R}_{L, \Sigma, \mathcal{E}, \varphi} \) maps an orthonormal basis of \( \text{Class}(W_G(L, \Sigma, \mathcal{E}), F) \), c.f., 2.3, onto an orthonormal basis of \( \text{Class}(G^F | [L, \Sigma, \mathcal{E}]) \), c.f., Theorem 3.4 and Remark 6.11.

7. Comparison Theorems for Character Sheaves

7.1. Assume \( \mathbf{M} \leq G \) is a Levi subgroup of \( G \) and \( (L, \Sigma, \mathcal{E}) \in \text{Cusp}(\mathbf{M}) \) is a cuspidal triple. We choose parabolic subgroups \( P \leq Q \leq G \) with Levi complements \( L \leq P \) and \( \mathbf{M} \leq Q \). In this section we note that analogues of Howlett–Lehrer’s Comparison Theorem, see [17, 5.9], hold for induction of character sheaves. Namely \( \text{ind}_{\mathbf{M} \leq Q}^G \mathcal{R}_{L, \Sigma, \mathcal{E}} \) corresponds to the usual induction \( \text{Ind}_{V_M(L, \Sigma, \mathcal{E})}^G \mathcal{R}_{L, \Sigma, \mathcal{E}, \varphi} \) under the correspondence described in 6.8. In the case where \( \Sigma \) contains a unipotent element this was pointed out by Lusztig in [25, 2.5]. Our observation is that this holds whenever the properties \( (\mathcal{R}_{L, \Sigma, \mathcal{E}}^G) \) and \( (\mathcal{R}_{M, \Sigma, \mathcal{E}}^M) \) hold.
7.2. In general if \( A \in \mathcal{M}(M) \) is an \( M \)-equivariant perverse sheaf then the complex \( \text{ind}^G_{M \subseteq Q}(A) \in \mathcal{D}_G(G) \) need not necessarily be perverse. However, if \( A \in \text{CSh}(M) \) then Lusztig has shown that \( \text{ind}^G_{M \subseteq Q}(A) \in \text{CSh}(G) \), see [24, 4.4]. Hence, we have \( \text{ind}^G_{M \subseteq Q} \) defines a \( \mathcal{O}_G \)-linear functor \( \text{CSh}(M) \to \text{CSh}(G) \) between abelian categories. In particular, we can appeal to the formalism discussed in Appendix B.

7.3. Let \( P \subseteq Q \subseteq G \) be parabolic subgroups of \( G \) with Levi complements \( L \subseteq P \) and \( M \subseteq Q \). We set \( \hat{P} = M \cap P \) which is a parabolic subgroup of \( M \) with Levi complement \( L \). By [24, 4.2, 4.4] we have an isomorphism

\[
\text{ind}^G_{M \subseteq Q}(\text{ind}^M_{L \subseteq P}(\mathcal{E})) \equiv \text{ind}^G_{L \subseteq P}(\mathcal{E}). \tag{7.4}
\]

After Theorem 5.5 this yields an isomorphism

\[
\text{ind}^G_{M \subseteq Q}(K^M_{L, \Sigma, \mathcal{E}}) \equiv K^G_{L, \Sigma, \mathcal{E}}
\]

and we obtain an algebra homomorphism \( \text{ind}^G_{M \subseteq Q} : A^M_{L, \Sigma, \mathcal{E}} \to A^G_{L, \Sigma, \mathcal{E}} \). We want to show the following compatibility.

**Proposition 7.5.** Assume properties \((p^G_{L, \Sigma, \mathcal{E}})\) and \((p^M_{L, \Sigma, \mathcal{E}})\) hold then we have a commutative diagram

\[
\begin{array}{ccc}
\eta^M_{L, \Sigma, \mathcal{E}} & \overset{\text{ind}^G_{M \subseteq Q}}{\longrightarrow} & \eta^G_{L, \Sigma, \mathcal{E}} \\
\downarrow & & \downarrow \\
\mathcal{Q}_\ell[W_M(L, \Sigma, \mathcal{E})] & \longrightarrow & \mathcal{Q}_\ell[W_G(L, \Sigma, \mathcal{E})]
\end{array}
\]

where the bottom arrow is the canonical inclusion of algebras and the vertical arrows are the isomorphisms described in Lemma 6.5.

**Proof.** For any object \( \square \) introduced in 4.1 we affix subscripts and superscripts, such as \( \square^G_{L \subseteq P} \), to indicate that it is defined with regards to \( \text{ind}^G_{L \subseteq P} \). Let \( D \in \mathcal{M}(\hat{X}^G_{L \subseteq P}) \) and \( D' \in \mathcal{M}(X^M_{L \subseteq P}) \) be the canonical perverse sheaves satisfying \( \hat{\eta}^G_{L \subseteq P} \mathcal{E} = \hat{\sigma}^G_{L \subseteq P} D \) and \( \hat{\eta}^M_{L \subseteq P} \mathcal{E} = \hat{\sigma}^M_{L \subseteq P} D' \) respectively. By definition we have

\[
\text{ind}^G_{L \subseteq P}(\mathcal{E}) = (\tau^G_{L \subseteq P})! D \quad \text{and} \quad \text{ind}^M_{L \subseteq P}(\mathcal{E}) = (\tau^M_{L \subseteq P})! D'.
\]

We have a well-defined equivariant morphism \( \lambda : \hat{X}^G_{L \subseteq P} \to \hat{X}^G_{M \subseteq Q} \) given by \( \lambda(g, hP) = (g, hQ) \) because \( P \subseteq Q \). In [24, 4.2(b)] Lusztig shows that \( \hat{\eta}^G_{M \subseteq Q}(\tau^M_{L \subseteq P})! D' = \hat{\sigma}^G_{M \subseteq Q} \lambda ! D \) so, again by definition, we have

\[
\text{ind}^G_{M \subseteq Q}(\text{ind}^M_{L \subseteq P}(\mathcal{E})) = (\tau^G_{M \subseteq Q})! \lambda ! D = (\tau^G_{M \subseteq Q} \circ \lambda !_D = (\tau^G_{L \subseteq P})! D = \text{ind}^G_{L \subseteq P}(\mathcal{E})
\]

because \( \tau^G_{L \subseteq P} = \tau^G_{M \subseteq Q} \circ \lambda \). Note that \( \sigma^G_{M \subseteq Q} \lambda _! D = \hat{\sigma}^G_{M \subseteq Q} \circ \lambda _! D \) and \( \hat{\sigma}^G_{M \subseteq Q}(\tau^M_{L \subseteq P})! D \) denotes the compactly supported pushforward of \( \hat{\eta}^G_{M \subseteq Q}(\tau^M_{L \subseteq P})! D \).

Now let \( \theta \) be an invertible endomorphism of \( \mathcal{E} \). If \( f \) and \( f' \) denote the unique
morphisms satisfying $\pi^G_{L \subseteq P}\theta = \delta^G_{L \subseteq P} f$ and $\pi^M_{L \subseteq P}\theta = \delta^M_{L \subseteq P} f'$ then by definition

$$\text{ind}^G_{L \subseteq P}(\theta) = (\pi^G_{L \subseteq P})_f$$

$$\text{ind}^M_{L \subseteq P}(\theta) = (\pi^M_{L \subseteq P})_f'.$$

An identical argument to that used by Lusztig shows that $\pi^G_{M \subseteq Q}(\pi^M_{L \subseteq P})_f = \delta^G_{M \subseteq Q}\lambda f$ and identically we get that $\text{ind}^G_{M \subseteq Q}(\text{ind}^M_{L \subseteq P}(\theta)) = \text{ind}^G_{L \subseteq P}(\theta)$. The statement now follows from Lemma 6.7 and the fact that the trivial module for $W_M(L, \Sigma, \delta')$ is the restriction for that of $W_G(L, \Sigma, \delta')$.

**Corollary 7.6.** Assume $M \leq G$ is a Levi subgroup and $(L, \Sigma, \delta') \in \text{Cusp}(M)$ is a cuspidal triple. If $(\mathcal{P}^M_{L, \Sigma, \delta'})$ and $(\mathcal{P}^G_{L, \Sigma, \delta'})$ hold then for any irreducible characters $\mu \in \text{Irr}(W_M(L, \Sigma, \delta'))$ and $\lambda \in \text{Irr}(W_G(L, \Sigma, \delta'))$ we have

$$\dim_{\mathbb{Q}_l} \text{Hom}_{\mathfrak{g}_G(G)}(K^G_\lambda, \text{ind}^G_{M \subseteq Q}(K^M_\mu)) = \langle \lambda, \text{Ind}^G_{W_M(L, \Sigma, \delta')}(\mu) \rangle_{W_G(L, \Sigma, \delta')}.$$

In particular, we have $K^G_\lambda | \text{ind}^G_{M \subseteq Q}(K^M_\mu)$ if and only if $\langle \lambda, \text{Ind}^G_{W_M(L, \Sigma, \delta')}(\mu) \rangle_{W_G(L, \Sigma, \delta')} \neq 0$.

**Proof.** By Corollary B.9 we have a $\overline{Q}_l$-linear isomorphism

$$\text{Hom}_{\mathfrak{g}_G(G)}(\text{ind}^M_{M \subseteq Q}(K^M_\mu), K^G_\lambda) \cong \text{Hom}_{\mathfrak{A}_G(G)}(A^G_{L, \Sigma, \delta'} \otimes A^M_{L, \Sigma, \delta'} \delta^M_{L, \Sigma, \delta'}(K^M_\mu), \delta^G_{L, \Sigma, \delta'}(K^G_\lambda)).$$

Using Proposition 7.5 we see that $A^G_{L, \Sigma, \delta'} \otimes A^M_{L, \Sigma, \delta'} \delta^M_{L, \Sigma, \delta'}(K^M_\mu)$ affords the induced character $\text{Ind}^G_{W_M(L, \Sigma, \delta')}(\mu)$. The statement now follows from the fact that the induced complex is semisimple.

**7.7.** Now assume $(L, \Sigma, \delta', \varphi) \in \text{Cusp}(M, F)$ is a rational cuspidal tuple then one could ask for similar compatibilities at the level of characteristic functions. The following gives the shadow of Corollary 7.6 at the level of functions.

**Proposition 7.8.** Assume $(L, \Sigma, \delta', \varphi) \in \text{Cusp}(M, F)$ is a fixed tuple. If $p$ is a good prime for $G$ and $Z(G)$ is connected then we have an equality

$$R^G_M \circ R^M_{L, \Sigma, \delta', \varphi} = R^G_{L, \Sigma, \delta', \varphi} \circ \text{Ind}^G_{W_M(L, \Sigma, \delta'), F}.$$

of linear maps $\text{Class}(W_M(L, \Sigma, \delta'), F) \to \text{Class}(G^F | [L, \Sigma, \delta'])$.

**Proof.** As remarked in Corollary A.4 we have the space $\text{Class}(W_M(L, \Sigma, \delta'), F)$ is spanned by the functions $\pi^M_{W_M(L, \Sigma, \delta'), F}$, with $x \in W_M(L, \Sigma, \delta'), F$, so it suffices to show the equality holds on these functions. By definition it is clear that

$$R^M_{Lw}(\chi^f_{\delta^f_{w, \varphi^f_{w}}}) = R^M_{L, \Sigma, \delta', \varphi}(\pi^{W_{M(L, \Sigma, \delta')}}_{w-1} F)$$

and by the transitivity of Deligne–Lusztig induction this yields

$$R^G_M \circ R^M_{Lw}(\chi^f_{\delta^f_{w, \varphi^f_{w}}}) = R^G_{Lw}(\chi^f_{\delta^f_{w, \varphi^f_{w}}}) = R^G_{L, \Sigma, \delta', \varphi}(\pi^{W_{M(L, \Sigma, \delta')}}_{w-1} F).$$

The equality now follows from (ii) of Proposition A.7.
8. The Case of a Split Levi Subgroup

8.1. Recall that if \( p \) is a good prime for \( G \) then to any irreducible character \( \chi \in \text{Irr}(G^F) \) one can associate its wave-front set \( \mathcal{O}_\chi^* \subseteq \mathcal{U}(G) \), which is an \( F \)-stable unipotent conjugacy class in \( G \), see [28, 11.2] and [34, 14.10]. If \( D_{G^F} : \text{Class}(G^F) \to \text{Class}(G^F) \) denotes Alvis–Curtis duality then we have a bijection \( \ast : \text{Irr}(G^F) \to \text{Irr}(G^F) \) defined by \( \chi^* = \pm D_{G^F}(\chi) \). This follows from the fact that \( D_{G^F} \) is an involutive isometry mapping characters to virtual characters, [11, §8]. The notions of wave-front set and unipotent support are related via the equality

\[
\mathcal{O}_\chi^* = \mathcal{O}_{\chi^*}.
\]

see [28, 11.2] and [34, 14.15].

Lemma 8.3. Assume \( p \) is a good prime for \( G \) then for any \( F \)-stable Levi subgroup \( M \subseteq G \) the following statements are equivalent:

(i) for any irreducible characters \( \eta \in \text{Irr}(M^F) \) and \( \chi \in \text{Irr}(G^F) \) satisfying \( \langle \chi, R_M^G(\eta) \rangle_{G^F} \neq 0 \) we have \( \mathcal{O}_\eta^* \subseteq \mathcal{O}_{\chi^*} \).

(ii) for any irreducible characters \( \eta \in \text{Irr}(M^F) \) and \( \chi \in \text{Irr}(G^F) \) satisfying \( \langle \chi, R_M^G(\eta) \rangle_{G^F} \neq 0 \) we have \( \mathcal{O}_\eta \subseteq \mathcal{O}_{\chi^*} \).

Proof. As \( p \) is a good prime for \( G \) we have that the Mackey formula holds by [7]. Hence \( R_M^G \circ D_{M^F} = D_{G^F} \circ R_M^G \) by [6, 10.13]. As \( D_{G^F} \) is an isometry we get that

\[
\langle \chi, R_M^G(\eta) \rangle_{G^F} = \langle D_{G^F}(\chi), D_{G^F}(R_M^G(\eta)) \rangle_{G^F} = \pm \langle \chi^*, R_M^G(\eta^*) \rangle_{G^F}.
\]

The equivalence now follows from (8.2) and the fact that \( \ast \) is a bijection. ■

8.4. Recall that a homomorphism of algebraic groups \( \iota : G \to \tilde{G} \) is called isotypic if the image \( \iota(G) \) contains the derived subgroup of \( \tilde{G} \) and the kernel \( \text{Ker}(\iota) \) is contained in the centre \( Z(G) \) of \( G \). We further assume that \( \tilde{G} \) is equipped with a Frobenius endomorphism \( F : \tilde{G} \to \tilde{G} \) and \( \iota \) is defined over \( F_q \), in the sense that \( \iota \circ F = F \circ \iota \). If \( \tilde{G} \) is a connected reductive algebraic group then for any Levi subgroup \( M \subseteq G \) of \( G \) we have \( \tilde{M} = \iota(M)Z(\tilde{G}) \subseteq \tilde{G} \) is a Levi subgroup of \( \tilde{G} \). The assignment \( M \mapsto \tilde{M} \) is a bijection between Levi subgroups sending \( F \)-stable Levi subgroups to \( F \)-stable Levi subgroups and \( (G,F) \)-split Levi subgroups to \( (\tilde{G},F) \)-split Levi subgroups.

8.5. We now wish to develop some lemmas which allow us to reduce checking the conditions in Lemma 8.3 to a suitably nice case. For this let us recall that an isotypic morphism \( \iota : G \to \tilde{G} \) is a regular embedding if \( \tilde{G} \) is a connected reductive algebraic group with \( Z(\tilde{G}) \) connected and \( \iota \) is a closed embedding. Given such a morphism we will implicitly identify \( G \) with a subgroup of \( \tilde{G} \) and identify \( \mathcal{U}(G) \) with \( \mathcal{U}(\tilde{G}) \). We note that we also have the notion of a smooth covering as defined in [35, 1.20].
Proposition 8.6. Let \( \iota : G \to \tilde{G} \) be a regular embedding and assume \( M \leq G \) and \( \tilde{M} \leq \tilde{G} \) are corresponding F-stable Levi subgroups. If \( M \) is \((G,F)\)-split, equivalently \( \tilde{M} \) is \((\tilde{G},\tilde{F})\)-split, then the conditions of Lemma 8.3 hold for the pair \((M,G)\) if and only if they hold for the pair \((\tilde{M},\tilde{G})\).

Proof. We first show that if (ii) of Lemma 8.3 holds for the pair \((\tilde{M},\tilde{G})\) then it holds for the pair \((M,G)\). Let \( \eta \in \text{Irr}(M^F) \) and \( \chi \in \text{Irr}(G^F) \) be irreducible characters satisfying \( \langle \chi, R_M^G(\eta) \rangle_{G^F} \neq 0 \). We choose an irreducible character \( \tilde{\eta} \in \text{Irr}(\tilde{M}^F) \) such that \( \text{Res}_{G^F}^G(\tilde{\eta}) = \eta + \lambda \) with \( \lambda \in \text{Class}(M^F) \) a character.

According to [6, 10.10] we have \( R_M^G \circ \text{Res}_{M^F}^M = \text{Res}_{G^F}^G \circ R_M^G \) which implies that

\[
\text{Res}_{G^F}^G(R_M^G(\tilde{\eta})) = R_M^G(\eta) + R_M^G(\lambda). \tag{8.7}
\]

As \( M \) is a \((G,F)\)-split Levi subgroup we have \( R_M^G \) is Harish-Chandra induction so the sum in (8.7) is a sum of characters. As \( \chi \) is a constituent of \( R_M^G(\eta) \) it is therefore also a constituent of \( \text{Res}_{G^F}^G(\tilde{\eta}) \). Hence we have

\[
\langle \chi, \text{Res}_{G^F}^G(R_M^G(\tilde{\eta})) \rangle_{G^F} = \langle \text{Ind}_{G^F}^G(\chi), \text{Res}_{G^F}^G(\tilde{\eta}) \rangle_{G^F} \neq 0.
\]

This implies there exists an irreducible character \( \tilde{\chi} \in \text{Irr}(\tilde{G}^F | R_M^G(\tilde{\eta})) \) such that \( \chi \in \text{Irr}(\tilde{G}^F | \text{Res}_{G^F}^G(\tilde{\chi})) \). In particular, we have \( \langle \tilde{\chi}, R_M^G(\tilde{\eta}) \rangle_{G^F} \neq 0 \) so \( 0_{\tilde{\chi}} \leq 0_{\chi} \) by assumption. The statement now follows from the fact that \( 0_{\chi} = 0_{\tilde{\chi}} \) and \( 0_{\eta} = 0_{\tilde{\eta}} \), see the proof of [13, Lemma 5.1].

Now assume (ii) of Lemma 8.3 holds for the pair \((M,G)\). Let \( \tilde{\eta} \in \text{Irr}(\tilde{M}^F) \) and \( \tilde{\chi} \in \text{Irr}(\tilde{G}^F) \) be irreducible characters satisfying \( \langle \tilde{\chi}, R_M^G(\tilde{\eta}) \rangle_{G^F} \neq 0 \). As \( M \) is \((G,F)\)-split we have \( R_M^G \) is Harish-Chandra induction and there exists a character \( \tilde{\lambda} \in \text{Class}(\tilde{G}^F) \) such that \( R_M^G(\tilde{\eta}) = \tilde{\chi} + \tilde{\lambda} \). Restricting we get

\[
R_M^G(\text{Res}_{M^F}^M(\tilde{\eta})) = R_M^G(\text{Res}_{G^F}^G(\tilde{\eta})) = \text{Res}_{G^F}^G(\tilde{\chi}) + \text{Res}_{G^F}^G(\tilde{\lambda}).
\]

Let \( \chi \in \text{Irr}(G^F | R_M^G(\tilde{\chi})) \) be an irreducible constituent of the restriction then there must exist an irreducible character \( \eta \in \text{Irr}(M^F | \text{Res}_{M^F}^M(\tilde{\eta})) \) such that \( \langle \chi, R_M^G(\eta) \rangle_{G^F} \neq 0 \). We now conclude that \( 0_{\tilde{\eta}} = 0_{\tilde{\eta}} \leq 0_{\chi} = 0_{\tilde{\chi}} \).

\[
\text{Lemma 8.8.} \text{ Let } \iota : \tilde{G} \to G \text{ be a surjective isotypic morphism with connected kernel. If } M \leq G \text{ and } \tilde{M} \leq \tilde{G} \text{ are any corresponding F-stable Levi subgroups then the conditions of Lemma 8.3 hold for the pair } (M,G) \text{ if they hold for the pair } (\tilde{M},\tilde{G}).
\]

Proof. As \( \text{Ker}(\iota) \) is connected we have \( \iota \) restricts to surjective homomorphisms \( \iota : \tilde{G}^F \to G^F \) and \( \iota : \tilde{M}^F \to M^F \) by the Lang–Steinberg theorem. We then have an isometry \( \text{Class}(G^F) \to \text{Class}(\tilde{G}^F) \) given by \( \chi \mapsto \chi \circ \iota \). In particular, for any irreducible characters \( \chi \in \text{Irr}(G^F) \) and \( \eta \in \text{Irr}(M^F) \) we have

\[
\langle R_M^G(\eta), \chi \rangle_{G^F} = \langle R_M^G(\eta), \chi \circ \iota \rangle_{G^F} = \langle R_M^G(\eta \circ \iota), \chi \circ \iota \rangle_{G^F}.
\]
where the last equality follows from [11, 13.22]. Now \( i \) restricts to a homeomorphism \( \iota(\tilde{G}) \to \iota(G) \) between the unipotent varieties and from the proof of [13, Lem. 5.2] we see that \( \iota(\emptyset_{\chi \eta}) = \emptyset_{\chi} \) and \( \iota(\emptyset_{\eta \chi}) = \emptyset_{\eta} \). Hence, if (ii) of Lemma 8.3 holds for the pair \((\tilde{M}, \tilde{G})\) then it holds for the pair \((M, G)\).

Proof. We show that the proof of [4, 2.6] can be used to show that (i) of Lemma 8.3 holds. Firstly, by a result of Asai, there exists a surjective isotypic morphism \( \iota \) see that for some character \( \chi \in \text{Irr}(\tilde{G}) \) and the results of Lusztig [28] are available to us, see [34, 13.6]. We may now proceed as [1.21]. By Lemma 8.8 we can thus assume that the derived subgroup \( \tilde{G} \) is connected and \([G, G] \leq G \) is simply connected. Now, applying Proposition 8.6 to the canonical regular embedding \([G, G] \to G\) we can assume that \( G \) is semisimple and simply connected.

With this assumption we have \( G = G_1 \times \cdots \times G_r \) with each \( G_i \leq G \) simple and simply connected. The Frobenius endomorphism \( F \) permutes the subgroups \( G_i \) and we may clearly assume that it does so transitively. Using [35, 8.3] it suffices to consider the case where \( G \) is simple and simply connected. Moreover, choosing a regular embedding \( G \to \tilde{G} \) we can assume that \( Z(G) \) is connected and \([G, G] \) is simply connected.

With this assumption we have \( G = \text{GL}_n(F) \).

Recall that \( \chi \in \text{Irr}(G^F) \) and \( \eta \in \text{Irr}(M^F) \) are such that \( \langle \chi, R^G_M(\eta) \rangle_{G^F} \neq 0 \). By the definition of the wave-front set there exists a unipotent element \( u \in \emptyset_{\eta}^F \) and a generalized Gelfand–Graev character (GGGGC) \( \Gamma^M_u \) of \( M \) such that \( \langle \eta, \Gamma^M_u \rangle_{M^F} \neq 0 \). Hence \( \Gamma^M_u = \eta + \lambda \) for some character \( \lambda \in \text{Class}(M^F) \) and \( R^G_M(\Gamma^M_u) = R^G_M(\eta) + R^G_M(\lambda) \) is a sum of characters, as \( M \) is assumed to be \((G,F)\)-split. In particular, as in the proof of Lemma 8.3, we must have

\[
0 \neq \langle \chi, R^G_M(\Gamma^M_u) \rangle_{G^F} = \langle D_{G^F}(\chi), R^G_M(D_{M^F}(\Gamma^M_u)) \rangle_{M^F}.
\]

This implies there exists an element \( v \in G^F \) such that \( \chi^*(v) \neq 0 \) and \( R^G_M(D_{M^F}(\Gamma^M_u))(v) \neq 0 \). Arguing as in the proof of [4, 2.6] we see that the second condition implies that \( v \) is unipotent and \( u^G \leq v^G \), see also [14, 24.(c)]. Moreover, the first condition implies that \( v \in \emptyset_{\chi^*} = \emptyset_{\chi} \), see [35, Theorem 1.26] or [1, Théorème 8.1]. Hence, we have \( \emptyset_{\eta}^* \leq \emptyset_{\chi}^* \).

9. Unipotent Supports of Characters and Character Sheaves

9.1. In this section we briefly recall partitions of \( \text{Irr}(CSh(G)) \) and \( \text{Irr}(G^F) \) defined by Lusztig. We introduce some notation that will be used in this and the subsequent
sections. For each \( w \in W_G = W_G(T_0) \) we choose a representative \( w' \in N_G(T_0) \) and set \( F w := F \circ t_w \). Let us denote by \( \mathcal{C}_G(T_0, F) \) the set of pairs \( (w, \theta) \) where \( w \in W_G \) and \( \theta \in \text{Irr}(T_0^w) \). The Weyl group \( W_G \) acts naturally on \( \mathcal{C}_G(T_0, F) \) by setting \( x \cdot (w, \theta) = (x w F(x^{-1}), \theta \circ t^{-1}_F) \). Moreover, to each pair \((w, \theta) \in \mathcal{C}_G(T_0, F)\) we have a corresponding Deligne–Lusztig virtual character \( R_G^w(\theta) \in \text{Class}(G^F) \) and the assignment \((w, \theta) \mapsto R_G^w(\theta)\) is constant on \( W_G \)-orbits.

**9.2.** Dually for each \( w \in W_G \) : \( W_G(T_0^w) \) we choose a representative \( w' \in N_G(T_0^w) \) and set \( F w := t_w \circ F \). Let us denote by \( \mathcal{D}_G(T_0, F) \) the set of pairs \( (w, s) \) such that \( w \in W_G = W_G(T_0^w) \) and \( s \in T_0^{wF} \). As above \( W_G \) acts naturally on \( \mathcal{D}_G(T_0, F) \) by setting \( x \cdot (w, s) = (x w F(x^{-1}), x s) \). Now, there is a natural anti-isomorphism \( * : W_G \to W_G \) induced by duality, which satisfies \( F^{-1}(w^*) = F(w)^* \). This can be extended to a bijection

\[
\delta : \mathcal{C}_G(T_0, F) \to \mathcal{D}_G(T_0^w, F)
\]

which is anti-equivariant in the sense that \( \delta(x \cdot (w, \theta)) = x^{* -1} \cdot \delta((w, \theta)) \).

**9.4.** For any \( s \in T_0^w \) and \( w^* \in W_G \) we define a class function \( R_G^w(s) \in \text{Class}(G^F) \) as follows. If \( s \not\in T_0^{wF} \) then we set \( R_G^w(s) = 0 \). Otherwise we set \( R_G^w(s) := R_G^w(\theta) \) where \((w, \theta) = \delta^{-1}((w^*, s)) \). If \( s \in T_0^w \) is a semisimple element then we associate to the corresponding \( W_G \)-orbit \((s) \in T_0^w / W_G \), a set

\[
\mathcal{E}(G^F, (s)) = \bigsqcup_{w^* \in W_G} \text{Irr}(G^F | R_G^{w^*}(s)).
\]

Note this set is empty unless the \( W_G \)-orbit of \( s \) is \( F \)-stable. A classic result of Deligne–Lusztig states that we have a partition

\[
\text{Irr}(G^F) = \bigsqcup_{(s) \in T_0^w / W_G} \mathcal{E}(G^F, (s)).
\]

**9.5.** We now consider a further refinement of this result. For this, let \( \text{Fam}(G^*, T_0^w) \) denote the set of all pairs \((s, \mathcal{C})\) where \( s \in T_0^w \) is a semisimple element and \( \mathcal{C} \) is a two-sided cell of \( W_G \cdot (s) \). We refer to the elements of \( \text{Fam}(G^*, T_0^w) \) as families. The Weyl group \( W_G \) acts naturally on \( \text{Fam}(G^*, T_0^w) \) by conjugation and we denote by \( \text{Fam}(G^*, T_0^w) \) the set of orbits under this action. To each family \( \mathcal{F} \in \text{Fam}(G^*, T_0^w) \) we have a corresponding unipotent class \( 0_G \subseteq \mathfrak{u}(G)/G \) of \( G \), see [28, 10.5] and [34, 12.9]. This assignment is invariant under the action of \( W_G \).

**9.6.** The Frobenius defines a permutation of \( \text{Fam}(G^*, T_0^w) \) and \( \text{Fam}(G^*, T_0^w) \). We denote by \( \text{Fam}(G^*, T_0^w)^F \) and \( \text{Fam}(G^*, T_0^w)^F \) the respective set of fixed points. By work of Lusztig we have decompositions

\[
\text{Irr}(G^F) = \bigsqcup_{\mathcal{F} \in \text{Fam}(G^*, T_0^w)^F} \mathcal{E}(G^F, \mathcal{F}), \quad \text{Irr}(\text{CSh}(G))^F = \bigsqcup_{\mathcal{F} \in \text{Fam}(G^*, T_0^w)^F} \text{Irr}(\text{CSh}(G), \mathcal{F}),
\]

see [28, 10.6, 11.1], [24, 16.7], and [34, 13.1, 14.7]. We note that if \( \mathcal{F} = (s, \mathcal{C}) \) then \( \mathcal{E}(G^F, \mathcal{F}) \subseteq \mathcal{E}(G^F, (s)) \). With this in place we have the following consequence of Shoji's
work.

**Lemma 9.7.** Assume $p$ is a good prime and $Z(G)$ is connected. If $A \in \text{Irr}(\text{CSh}(G))^F$ is an $F$-stable character sheaf and $\chi \in \text{Irr}(G^F)$ is an irreducible character satisfying $\langle \chi, \chi_A \rangle_{G^F} \neq 0$ then $\mathcal{O}_A = \mathcal{O}_\chi$.

**Proof.** A weak form of Shoji’s result [31] states that for any family $\mathcal{F} \in \text{Fam}(G^+, T_0^\mathbb{Z})$ the subspace of $\text{Class}(G^F)$ spanned by $\mathcal{E}(G^F, \mathcal{F})$ coincides with that spanned by $\{\chi_A | A \in \text{Irr}(\text{CSh}(G), \mathcal{F})\}$. The statement now follows from the fact that if $\chi \in \mathcal{E}(G^F, \mathcal{F})$ then $\mathcal{O}_\chi = 0$, see [28, 11.2] and [16, §3.C], and if $A \in \text{Irr}(\text{CSh}(G), \mathcal{F})$ then $\mathcal{O}_A = 0$, see [28, 10.7] and [34, 13.8].

10. **Proof of Theorems 1.25 and 1.29**

**Proof (of Theorem 1.29).** Let $K = \text{ind}_{M \subseteq Q}^G(A)$. As in the proof of [24, 15.2] we will assume that $F_1 : G \to G$ is a Frobenius endomorphism such that $M$ and $Q$ are $F_1$-stable. Moreover, by possibly replacing $F_1$ by a power, we can assume that there exist isomorphisms $F_1^* A \to A$ and $F_1^* B \to B$. In particular, we have an induced isomorphism $F_1^* K \to K$ as in Lemma 4.4. As $K \in \text{CSh}(G)$ is semisimple we have by Lemmas 4.4 and 4.8 that

$$\langle \chi_B, R^G_{M \subseteq Q}(\chi_A) \rangle_{G^F} = \text{Tr}(\sigma_B, \sigma_K(B)) \neq 0.$$ 

Here $R^G_{M \subseteq Q}$ denotes Harish-Chandra induction with respect to the Frobenius endomorphism $F_1$ and the $F_1$-stable parabolic subgroup $Q$.

As the irreducible characters form an orthonormal basis of the space of class functions we have decompositions

$$\chi_A = \sum_{\eta \in \text{Irr}(M^{F_1})} \langle \eta, \chi_A \rangle_{M^{F_1}} \eta, \quad R^G_M(\eta) = \sum_{\chi \in \text{Irr}(G^{F_1})} \langle \chi, R^G_M(\eta) \rangle_{G^{F_1}} \chi,$$

where $\eta \in \text{Irr}(M^{F_1})$. In particular, we have

$$\langle \chi_B, R^G_M(\chi_A) \rangle_{G^{F_1}} = \sum_{\eta \in \text{Irr}(M^{F_1})} \sum_{\chi \in \text{Irr}(G^{F_1})} \langle \eta, \chi_A \rangle_{M^{F_1}} \cdot \langle \chi, R^G_M(\eta) \rangle_{G^{F_1}} \cdot \langle \chi_B, \chi \rangle_{G^{F_1}}.$$

As $\langle \chi_B, R^G_M(\chi_A) \rangle_{G^{F_1}} \neq 0$ there must exist irreducible characters $\eta \in \text{Irr}(M^{F_1})$ and $\chi \in \text{Irr}(G^{F_1})$ such that

$$\langle \eta, \chi_A \rangle_{M^{F_1}} \neq 0, \quad \langle \chi, R^G_M(\eta) \rangle_{G^{F_1}} \neq 0, \quad \text{and} \quad \langle \chi_B, \chi \rangle_{G^{F_1}} \neq 0.$$

By Theorem 8.9 we must have $\mathcal{O}_\eta \leq \mathcal{O}_\chi$ but by Lemma 9.7 $\mathcal{O}_A = \mathcal{O}_\eta$ and $\mathcal{O}_B = \mathcal{O}_\chi$, which yields the statement.

**Proof (of Theorem 1.25).** Interchanging the roles of the two bases in the above argu-
ment we get a decomposition
\[
\langle \chi, R^G_M(\eta) \rangle_{G^F} = \sum_{A \in \text{Irr}(\text{Csh}(M))^F} \sum_{B \in \text{Irr}(\text{Csh}(G))^F} \langle \chi_A, \eta \rangle_{M^F} \cdot \langle \chi_B, R^G_M(\chi_A) \rangle_{G^F} \cdot \langle \chi, \chi_B \rangle_{G^F}.
\]
Hence, if \( \langle \chi, R^G_M(\eta) \rangle_{G^F} \neq 0 \) then there exist \( F \)-stable sheaves \( A \in \text{Irr}(\text{Csh}(M))^F \) and \( B \in \text{Irr}(\text{Csh}(G))^F \) such that
\[
\langle \eta, \chi_A \rangle_{L^F} \neq 0, \quad \langle \chi_B, R^G_M(\chi_A) \rangle_{G^F} \neq 0, \quad \text{and} \quad \langle \chi, \chi_B \rangle_{G^F} \neq 0.
\]

We can assume that \((L, \Sigma, \sigma, \varphi) \in \text{Cusp}(M, F)\) is a cuspidal tuple such that \( A \in \text{Irr}(\text{Csh}(M) | K^M_{L, \Sigma, \sigma})^F\). There then exists an irreducible character \( \lambda \in \text{Irr}(W_M(L, \Sigma, \sigma)) \) and extension \( \tilde{\lambda} \in \text{Irr}(W_M(L, \Sigma, \sigma):F) \) such that \( A \cong K^M_{L, \Sigma, \sigma} \) and \( \chi_A = R^M_{L, \Sigma, \sigma, \varphi}(\tilde{\lambda}) \), see Lemma 6.15. As \( \langle \chi_B, R^G_M(\chi_A) \rangle_{G^F} \neq 0 \) it follows from Proposition 7.8 that there exists an irreducible character \( \mu \in \text{Irr}(W_G(L, \Sigma, \sigma)) \) and extension \( \tilde{\mu} \in \text{Irr}(W_G(L, \Sigma, \sigma):F) \) such that \( B \cong K^G_{L, \Sigma, \sigma} \) and \( \chi_B = R^G_{L, \Sigma, \sigma, \varphi}(\tilde{\mu}) \), and
\[
\langle \tilde{\mu}, \text{Ind}^{W_G(L, \Sigma, \sigma, \varphi)}_{W_M(L, \Sigma, \sigma, \varphi)}(\tilde{\lambda}) \rangle_{W_G(L, \Sigma, \sigma, \varphi):F} \neq 0,
\]
because \( R^G_{L, \Sigma, \sigma, \varphi} \) is an isometry onto its image, c.f., Lemma 6.15.

By Lemma 2.7 we must have \( \langle \mu, \text{Ind}^{W_G(L, \Sigma, \sigma, \varphi)}_{W_M(L, \Sigma, \sigma, \varphi)}(\lambda) \rangle_{W_G(L, \Sigma, \sigma, \varphi):F} \neq 0 \) which implies that \( B \mid \text{Ind}^G_M(A) \) by Corollary 7.6. However, applying Theorem 1.29 and Lemma 9.7 we get that
\[
\Theta_\eta = \Theta_A \leq \Theta_B = \Theta_{\chi},
\]
and so we are done by Lemma 8.3.

11. Bounding the Number of Constituents in a Lusztig Restriction

11.1. The goal of this section is to obtain an easy bound on \( |\text{Irr}(M^F | * R^G_M(\chi))| \) for any irreducible character \( \chi \in \text{Irr}(G^F) \). To do this we use the partitioning of \( \text{Irr}(G^F) \) into Lusztig series described in Section 9. With regards to this we associate to any element \( s \in T_0^* \) the set
\[
T_{W_G^*}(s, F) = \{ w \in W_G^* | wF(s) = s \}.
\]
Either this set is empty or it is a coset of the form \( W_G^*(s)w \). With this we have the following, which is in the spirit of an argument of Lusztig [21].

Lemma 11.2. For any semisimple element \( s \in T_0^* \) we have
\[
|E(G^F, (s))| \leq |W_G^*(s)|^2.
\]

Proof. We start by noting that for any pair \( (w, \theta) \in E_G(T_0, F) \) the Mackey formula for tori implies that
\[
\langle R^G_w(\theta), R^G_w(\theta) \rangle = |\text{Stab}_W(w, \theta)|,
\]
see [11, 11.15]. Here $\text{Stab}_W(w, \theta)$ is the stabiliser of the pair under the $W$-action mentioned in 9.1. Therefore, if $\delta([w, \theta]) = (w^*, s)$, where $\delta$ is the bijection in (9.3), then we have

$$|\text{Irr}(G^F | R^G_W(\theta))| \leq |\text{Stab}_W(w, \theta)| = |\text{Stab}_{W^*}(w^*, s)| \leq |W_{G^*}(s)|$$

by the anti-equivariance of $\delta$. It now suffices to observe that $\text{E}(G^F, (s))$ is a union of the sets $\text{Irr}(G^F | R^G_W(\theta))$. Moreover, the non-empty such sets are indexed by $T_{W^*}(s, F)$ whose cardinality is $W_{G^*}(s)$. \hfill $\blacksquare$

**Corollary 11.3.** Assume $s \in T_0^*$ is a semisimple element and $\chi \in \text{E}(G^F, (s))$ is an irreducible character then for any Levi subgroup $M \leq G$ we have

$$|\text{Irr}(M^F | \ast R^G_M(\chi))| \leq |W_{G^*}(s)|^2.$$

**Proof.** We will assume our torus $T_0^*$ and dual Levi $M^*$ chosen such that $T_0^* \leq M^*$. By [6, 11.11] we have

$$\text{Irr}(M^F | \ast R^G_M(\chi)) \leq \bigcup_{(t) \subseteq T_0^*/W_{M^*}} \text{E}(M^F, (t)),$$

where $(s) \in T_0^*/W_{G^*}$ is the $W_{G^*}$-orbit of $s$. By Lemma 11.2 each series $\text{E}(M^F, (t))$ is bounded by $|W_{M^*}(t)|^2$. However, there are at most $|W_{G^*}(t)|/|W_{M^*}(t)|$ number orbits in the above sum. The bound now follows easily as $|W_{M^*}(t)| \leq |W_{G^*}(t)| = |W_{G^*}(s)|$. \hfill $\blacksquare$

12. **Bounding the Multiplicities**

**Proposition 12.1.** Assume $p$ is a good prime for $G$ and $Z(G)$ is connected. If $B_1, B_2 \in \text{Irr}(\text{CSh}(M))^F$ are $F$-stable character sheaves and $(L, \Sigma, \epsilon) \in \text{Cusp}(M)$ is a cuspidal triple such that the set $\text{Irr}(\text{CSh}(M) | K^M_{L, \Sigma, \epsilon})$ contains either $B_1$ or $B_2$ then we have

$$|\langle R^G_M(X_{B_1}), R^G_M(X_{B_2}) G^F \rangle| \leq |W_G(L, \Sigma, \epsilon)|.$$

**Proof.** Let us fix a cuspidal tuple $(L_1, \Sigma_1, \epsilon_1, \varphi_1) \in \text{Cusp}(M, F)$ such that $B_1 \in \text{Irr}(\text{CSh}(M))^F | K^M_{L_1, \Sigma_1, \epsilon_1})$. After Lemma 6.15 we have $X_{B_1} = \tau^M_{L_1, \Sigma_1, \epsilon_1, \varphi_1}(\hat{\lambda}_1)$ for some irreducible character $\hat{\lambda}_1 \in \text{Irr}(W_M(L_1, \Sigma_1, \epsilon_1) : F \downarrow W_M(L_1, \Sigma_1, \epsilon_1))$. If $\langle R^G_M(X_{B_1}), R^G_M(X_{B_2}) G^F \rangle$ is zero then there is nothing to show so we will assume that this inner product is non-zero. If this is the case then after Lemma 6.15 and Proposition 7.8 we must have the tuples $(L_1, \Sigma_1, \epsilon_1)$ and $(L_2, \Sigma_2, \epsilon_2)$ are in the same $G$-orbit. We fix a representative $(L, \Sigma, \epsilon) \in \text{Cusp}(M)^F$ of that $G$-orbit.

For brevity let us set $W_G = W_G(L, \Sigma, \epsilon)$ and $W_M = W_M(L, \Sigma, \epsilon)$. There exists an element $g_1 \in G$ such that $(L_1, \Sigma_1, \epsilon_1) = (L, \Sigma, \epsilon) g_1^{-1} \in N_G(L, \Sigma, \epsilon)$ represents an element $w_1^{-1} \in W_G$. Conjugating by $g_1$ identifies the pair $(W_M(L_1, \Sigma_1, \epsilon_1), F)$ with $(W_M(w_1 F), \tilde{\lambda}_1)$. Identifying $\tilde{\lambda}_1$ as an irreducible character of $W_M : w_1 F$, c.f., Section 2, we get from Proposition 7.8 that

$$\langle R^G_M(X_{B_1}), R^G_M(X_{B_2}) G^F \rangle = \langle \text{Ind}_{W_M(w_1 F)}^{W_G(w_1 F)}(\tilde{\lambda}_1), \text{Ind}_{W_M(w_2 F)}^{W_G(w_2 F)}(\tilde{\lambda}_2) \rangle_{W_G(F)}.$$
We now apply Corollary 2.8.

**Proposition 12.2.** Assume $p$ is a good prime for $G$ and $Z(G)$ is connected then for any irreducible character $\eta \in \operatorname{Irr}(M^F)$ we have

$$|\langle R^G_M(\eta), R^G_M(\eta) \rangle_{G^F}| \leq B(M)^4 \cdot |W_G|.$$  

**Proof.** Expanding $\eta$ in terms of characteristic functions of character sheaves we obtain a decompositon

$$\langle R^G_M(\eta), R^G_M(\eta) \rangle_{G^F} = \sum_{B_1, B_2 \in \operatorname{Irr}(C\operatorname{Sh}(M)^F)} \langle \chi_{B_1}, \eta \rangle_{M^F} \cdot \langle \chi_{B_2}, \eta \rangle_{M^F} \cdot \langle R^G_M(\chi_{B_1}), R^G_M(\chi_{B_2}) \rangle_{G^F}.$$  

Let us assume that $\mathcal{F} \in \operatorname{Fam}(M^*, T^*_g)^F$ is a family such that $\eta \in \mathcal{E}(M^F, \mathcal{F})$, c.f., the proof of Lemma 9.7. Moreover, let us denote by $\operatorname{Class}(M^F, \mathcal{F}) \subseteq \operatorname{Class}(M^F)$ the subspace spanned by the irreducible characters in $\mathcal{E}(M^F, \mathcal{F})$.

Note that for any character sheaf $B \in \operatorname{Irr}(C\operatorname{Sh}(M)^F)$ we have

$$1 = \langle \chi_{B}, \chi_{B} \rangle_{M^F} = \sum_{\eta' \in \operatorname{Irr}(M^F)} \langle \eta', \chi_{B} \rangle_{M^F} \cdot \langle \eta', \chi_{B} \rangle_{M^F} = \sum_{\eta' \in \operatorname{Irr}(M^F)} |\langle \eta', \chi_{B} \rangle_{M^F}|$$

by the orthonormality of the irreducible characters of $M^F$ and the characteristic functions of the character sheaves, c.f., Theorem 3.4. This implies that $|\langle \eta, \chi_{B} \rangle| \leq 1$.

In [22, Chapter 4] Lusztig has associated to the family $\mathcal{F}$ a pair $(\mathcal{G}_\mathcal{F}, \phi)$ consisting of a finite group $\mathcal{G}_\mathcal{F}$ and an automorphism $\phi \in \operatorname{Aut}(\mathcal{G}_\mathcal{F})$. Moreover, he has defined a corresponding set $\overline{M}(\mathcal{G}_\mathcal{F}, \phi)$ consisting of pairs $(x, \sigma)$, where $x \in \mathcal{G}_\mathcal{F}$ and $\sigma \in \operatorname{Irr}(C_{\mathcal{G}_\mathcal{F}}(x))$, taken up to equivalence modulo the action of $\mathcal{G}_\mathcal{F}: \phi$ defined by $g \cdot (x, \sigma) = (g x, \sigma \circ \iota_g^{-1})$. Note that $\mathcal{G}_\mathcal{F}: \phi$ is a coset as in Section 2 and $C_{\mathcal{G}_\mathcal{F}}(x)$ denotes the stabiliser of $x$ under the natural conjugation action of $\mathcal{G}_\mathcal{F}$ on $\mathcal{G}_\mathcal{F}: \phi$. The main result of [22] shows that there is a bijection

$$\overline{M}(\mathcal{G}_\mathcal{F}, \phi) \to \mathcal{E}(G^F, \mathcal{F}).$$

By Shoji’s result [31] we see that if $B \in \operatorname{Irr}(C\operatorname{Sh}(M)^F)$ is a character sheaf satisfying $\langle \eta, \chi_{B} \rangle \neq 0$ then $\chi_{B} \in \operatorname{Class}(G^F, \mathcal{F})$. Moreover, as the characteristic functions form a basis there can be at most

$$\dim \operatorname{Class}(G^F, \mathcal{F}) = |\overline{M}(\mathcal{G}_\mathcal{F}, \phi)| \leq |\mathcal{G}_\mathcal{F}|^2$$

character sheaves satisfying $\langle \eta, \chi_{B} \rangle \neq 0$. Combining this with Proposition 12.1 we get that

$$\langle R^G_M(\eta), R^G_M(\eta) \rangle \leq |\mathcal{G}_\mathcal{F}|^4 \cdot |W_G(L, \Sigma, \mathcal{E})|$$

where $(L, \Sigma, \mathcal{E}) \in \operatorname{Cusp}(M)$ is a tuple as in the statement of Proposition 12.1.

Now $W_G(L, \Sigma, \mathcal{E})$ is a subgroup of the relative Weyl group $W_G(L)$ which may be identified with a section of the Weyl group $W_G$. In particular we have $|W_G(L, \Sigma, \mathcal{E})| \leq |W_G|$. It is known that there exists an element $g \in M^*$ in a group dual to $M$ such
that the group $G_\gamma$ can be identified with a quotient of the component group $A_{\mathcal{M}^*}(g) = C_{\mathcal{M}^*}(g)/C_{\mathcal{M}}(g)$, see [22, Chapter 13] and [30]. If $g = su = us$ is the Jordan decomposition of the element then we have $A_{\mathcal{M}^*}(g) \cong A_{C_{\mathcal{M}^*}(s)}(u)$.

Moreover, if $\pi: C_{\mathcal{M}^*}(s) \to H := \tilde{C}_{\mathcal{M}^*}(s) := C_{\mathcal{M}^*}(s)/Z(C_{\mathcal{M}^*}(s))$ is the natural quotient map then we have $\pi$ defines an isomorphism $A_{C_{\mathcal{M}^*}(s)}(u) \cong A_H(\pi(u))$. If $\gamma: H_{sc} \to H$ is a simply connected cover of $H$ and $v \in \mathcal{U}(H_{sc})$ is the unique unipotent element satisfying $\gamma(v) = \pi(u)$ then $\gamma$ defines a surjective homomorphism $A_{H_{sc}}(v) \to A_H(\pi(u))$. This shows that $|G_\gamma| \leq |A_{H_{sc}}(v)| \leq B(M)$. ■

12.3. Next we record a lemma that is used in the proof of Theorem 1.7, which is essentially observed in [11, 12.22] and [6, §25.A]. We include a proof of this result for the convenience of the reader.

**Lemma 12.4.** Let $M \leq G$ be an $F$-stable Levi subgroup. If $g \in M^F$ is any element satisfying $C_M^G(g) \leq M$ then we have $\chi(g) = ^*R_M^G(\chi)(g)$ for any irreducible character $\chi \in \text{Irr}(G^F)$.

**Proof.** Let $g = su = us$ be the Jordan decomposition of the element. As $C_M^G(g) \leq M$ we have by [5, 1.3] that $C_M^G(s) \leq M$ so $C_M^G(s) = C_M^G(s)$. Thus, by the formula in [11, 12.5]

$$ \text{Res}_{C_M^G(s)}^M \circ ^*R_M^G = ^*R_M^G \circ \text{Res}_{C_M^G(s)}^G = \text{Res}_{C_M^G(s)}^G. $$

Note that $g \in C_M^G(s) \circ \pi[11, \text{Prop. 2.5}]$. Hence the statement follows by evaluating this formula at $\chi$ and then further at $g$. ■

**Proof (of Corollary 1.10).** We assume $\iota: G \to \tilde{G}$ is a regular embedding and $\tilde{M} \leq \tilde{G}$ is the $F$-stable Levi subgroup corresponding to $M \leq G$, c.f., 8.4. Consider an irreducible character $\tilde{\chi} \in \text{Irr}(\tilde{G}^F)$ such that $\chi$ is a constituent of the restriction $\text{Res}_{\tilde{G}^F}^F(\tilde{\chi})$. By Theorem 1.7 we have

$$ |\tilde{\chi}(g)| \leq f(r) \cdot \tilde{\chi}(1)^{\alpha_G(M,F)}. $$

As $\iota$ defines a bijection between the unipotent classes of $G$ and $\tilde{G}$ which preserves the dimension of each class, and similarly for $M$ and $\tilde{M}$, we have $\alpha_G(\tilde{M}, F) = \alpha_G(M, F)$. By a result of Lusztig [26] the restriction

$$ \text{Res}_{\tilde{G}^F}^F(\tilde{\chi}) = \chi_1 + \ldots + \chi_m, $$

with each $\chi_i \in \text{Irr}(G^F)$, is multiplicity free. Hence, if $\chi$ is $\tilde{G}^F$-invariant, then $\chi = \text{Res}_{\tilde{G}^F}^F(\tilde{\chi})$ and we are done in this case.

Next assume that $g\tilde{G}^F = g\tilde{M}$. This implies that $\tilde{G}^F = C_{\tilde{G}^F}(g)G^F$, and so we can find $x_i \in C_{\tilde{G}^F}(g)$ such that $x_i = \chi_1$. It follows that $\chi_i(g) = \chi(g)$ and $\tilde{\chi}(g) = m\chi(g)$. Since $\tilde{\chi}(1) = m\chi(1)$, we now have

$$ |\chi(g)| = |\tilde{\chi}(g)|/m \leq f(r)\tilde{\chi}(1)^{\alpha_G(M,F)}/m \leq f(r)\chi(1)^{\alpha_G(M,F)}, $$

as claimed.
Suppose now that $C_G(g)$ is connected, and consider any $h \in g^{G^F} \subseteq G^F$. As $\tilde{G} = Z(\tilde{G})G$, we have that $h \in G^F$ is $G$-conjugate to $g$. By the Lang-Steinberg theorem, $h$ is $G^F$-conjugate to $g$. Thus $g^{G^F} = g^{G^F}$, and we are done by the previous paragraph.

Finally, if $[G,G]$ is simply connected and $g$ is semisimple, then $C_G(g)$ is connected by Steinberg’s theorem, and we can apply the previous result. ■

We end by recording the following observation:

**Lemma 12.5.** For an $F$-stable Levi subgroup $M \subseteq G$ and an element $g \in M^F$ with semisimple part $s$, consider the following four conditions:

(i) $C_G^0(s) \subseteq M$,

(ii) $C_G^0(g) \subseteq M$,

(iii) $C_G^0(s)^F \subseteq M^F$,

(iv) $C_G^0(g)^F \subseteq M^F$.

Then (i) and (ii) are equivalent. Moreover, if $M$ is $(G,F)$-split then all four conditions are equivalent.

**Proof.** The equivalence of (i) and (ii) is part of [5, Lemma 1.2]. Assume now that $M$ is $(G,F)$-split. Certainly, (i) ⇒ (iii) ⇒ (iv). We now prove that (iv) ⇒ (i). Let $Q \leq G$ be an $F$-stable parabolic subgroup with Levi complement $M$. We denote by $V$ the unipotent radical of $Q$ so that $Q = M \ltimes V$; note that $V$ is $F$-stable because $Q$ is. By a result of Spaltenstein $C_V(g)$ is connected, see [5, 1.2], so $C_V(g) \leq C_G^0(g)$. Hence, (iv) implies that $C_V(g)^F \subseteq M \cap V = \{1\}$. However, $C_V(g)$ is a connected group all of whose elements are unipotent, so by Rosenlicht’s Theorem we have that

$$|C_V(g)| = q^{\dim C_V(g)}.$$

This implies that $\dim C_V(g) = 0$, so by [5, 1.3] we have that $C_G^0(s) \subseteq M$. ■

**A. Functions on $G$-sets**

**A.1.** Assume $G$ is a finite group and $(X, \alpha)$ is a finite $G$-set, i.e., $X$ is a finite set equipped with an action map $\alpha : G \times X \to X$. For each $g \in G$ we denote by $\alpha_g = \alpha(g, -) : X \to X$ the corresponding permutation of $X$. We denote by $\text{Orb}_G(x)$ and $\text{Stab}_G(x)$ the orbit and stabiliser of $x \in X$ respectively. Moreover, the set of all orbits will be denoted by $X/G$.

**A.2.** We will be interested in the finite dimensional $\overline{Q}_t$-vector space $\text{Fun}(X)$ of functions $f : X \to \overline{Q}_t$. The space is equipped with a form $\langle -,- \rangle_X : \text{Fun}(X) \times \text{Fun}(X) \to \overline{Q}_t$ defined by

$$\langle f, f' \rangle_X = \frac{1}{|G|} \sum_{x \in X} f(x)f'(x).$$
We then have a subspace of functions invariant under the $G$-action

$$\text{Fun}_G(X) = \{ f \in \text{Fun}(X) \mid f \circ \alpha_g = f \text{ for all } g \in G \}.$$  

The form $(-,-)_X$ restricts to a form on $\text{Fun}_G(X)$. For each element $x \in X$ we define a function $\pi^X_x \in \text{Fun}(X)$ by setting

$$\pi^X_x(y) = \begin{cases} \left| \text{Stab}_G(x) \right| & \text{if } y \in \text{Orb}_G(x) \\ 0 & \text{otherwise}. \end{cases}$$

The following is easy.

**Lemma A.3.** For any $x \in X$ we have $\pi^X_x \in \text{Fun}_G(X)$ and if $f \in \text{Fun}_G(X)$ then

$$f = \frac{1}{|G|} \sum_{x \in X} f(x) \pi^X_x.$$  

In particular, we have $f(x) = \langle f, \pi^X_x \rangle_X$.

**Corollary A.4.** If $x_1, \ldots, x_r \in X$ are representatives for the orbits $X/G$ then the functions $\pi^X_{x_1}, \ldots, \pi^X_{x_r}$ form an orthogonal basis of $\text{Fun}_G(X)$. In particular, $\dim_{Q_f} (\text{Fun}_G(X)) = |X/G|$.

**A.5.** Assume now that $H \leq G$ is a subgroup and $(Y, \beta)$ is an $H$-set. A function $\psi : Y \to X$ is an $H$-map if it is equivariant with respect to the actions, i.e., $\alpha_h \circ \psi = \psi \circ \beta_h$ for all $h \in H$. To each $H$-map $\psi : Y \to X$ we have a corresponding restriction map $\psi^* : \text{Fun}_G(X) \to \text{Fun}_H(Y)$ defined by $\psi^*(f) = f \circ \psi$. We can also define an induction map $\psi_* : \text{Fun}_H(Y) \to \text{Fun}_G(X)$ by setting

$$\psi_*(f)(y) = \frac{1}{|H|} \sum_{(g,y) \in G \times Y} f(y).$$

**Remark A.6.** If $\psi$ is injective then the formula for induction takes on the following more familiar form

$$\psi_*(f)(x) = \frac{1}{|H|} \sum_{g \in G} f(\psi^{-1}(\alpha_g(x))).$$

Moreover, if we take $\alpha = \iota$ to be the conjugation action then $\text{Fun}_G(G) = \text{Class}(G)$ is the usual space of $\mathbb{Q}_f$-class functions. If $H \leq G$ is a subgroup then $(H, \iota)$ is an $H$-set and taking $\psi : H \to G$ to be the natural inclusion map we get that the induction map $\psi_*$ is the usual induction on the space of class functions.

**Proposition A.7.** Assume $G$ is a finite group and $H \leq G$ is a subgroup. Moreover, let $(X, \alpha)$ be a $G$-set and $(Y, \beta)$ be an $H$-set. If $\psi : Y \to X$ is an $H$-map then the following statements hold:

(i) for any $f \in \text{Fun}_H(Y)$ and $f' \in \text{Fun}_G(X)$ we have $\langle \psi_*(f), f' \rangle_X = \langle f, \psi^*(f') \rangle_Y$,

(ii) for any $y \in Y$ we have $\psi_*(\pi^Y_y) = \pi^X_{\psi(y)}$. 
(iii) if \( K \leq H \) is a subgroup and \( Z \) is a \( K \)-set then for any \( K \)-map \( \lambda : Z \to Y \) we have 
\[
(\psi \circ \lambda)^* = \lambda^* \circ \psi^* \quad \text{and} \quad (\psi \circ \lambda)_* = \psi_* \circ \lambda_*.
\]

**Proof.** (i). We have 
\[
\langle \psi_*(f), f' \rangle_X = \frac{1}{|G|} \sum_{x \in X} \psi_*(f)(x)f'(x) = \frac{1}{|G||H|} \sum_{(g, x, y) \in G \times X \times Y} \alpha_g(x) = \psi(y) f(y)f'(x)
\]
As \( f'(x) = f'(\alpha_g(x)) = f'(\psi(y)) \) the right hand side depends only on \( y \). The result now follows from observing that 
\[
|(\{g, x \in G \times X \mid \alpha_g(x) = \psi(y)\}| = |G|.
\]

(ii). It is clear from the definition of induction that 
\[
\langle \psi_*(\pi^X_y), \pi^X_x \rangle_X = \psi_*(\pi^X_{\psi(y)})(x) \text{ is 0 unless } x \in \text{ Orb}_G(\psi(y)).
\]
By (i) we get 
\[
\langle \psi_*(\pi^X_{\psi(y)}), \pi^X_{\psi(y)} \rangle_X = \langle \pi^X_y, \psi^*(\pi^X_{\psi(y)})(y) = \psi^*(\pi^X_{\psi(y)})(y) = |\text{Stab}_G(\psi(y))|
\]
and the result follows.

(iii). The fact that \( (\psi \circ \lambda)^* = \lambda^* \circ \psi^* \) is obvious. By (ii) we have 
\[
(\psi \circ \lambda)_*(\pi^Z_z) = \pi^Z_{\psi(\lambda(z))} = (\psi_* \circ \lambda_*)(\pi^Z_z)
\]
for any \( z \in Z \) so we must have \( (\psi \circ \lambda)_* = \psi_* \circ \lambda_* \).

### B. Decomposing Semisimple Objects in Abelian Categories

From now until the end of this section we assume that \( \mathcal{A} \) is a locally finite \( k \)-linear abelian category, where \( k = \overline{k} \) is an algebraically closed field, and \( K \in \mathcal{A} \) is a fixed semisimple object.

#### B.1. We refer to [12, Chapter 1] for the basic definitions concerning abelian categories. Recall that an object \( A \in \mathcal{A} \) is said to be a summand of \( K \) if there exists a pair of morphisms

\[
A \xrightarrow{m} K
\]

such that \( pm = \text{Id}_A \). Note we necessarily have \( p \) is an epimorphism, \( m \) is a monomorphism, and \( u = mp \) is an idempotent. Moreover, if \( B = \text{Ker}(u) \) then there exist morphisms

\[
B \xrightarrow{q} K
\]

such that \( q_1 = \text{Id}_B \) and \( mp + tq = \text{Id}_K \). In other words, we have \( K \cong A \oplus B \). We write \( A \mid K \) to indicate that \( A \) is a summand of \( K \).

#### B.2. As \( K \) is assumed to be semisimple there exist finitely many simple objects
such that $m_1p_1 + \cdots + m_rp_r = \text{Id}_K$ and $p_im_j = \delta_{i,j} \text{Id}_{A_j}$, where $\delta_{i,j}$ is the Kronecker delta. We usually write $K = A_1 \oplus \cdots \oplus A_r$ to indicate it is a direct sum. As the Krull-Schmidt theorem holds in $\mathcal{A}$ we have the following.

**Lemma B.3.** If $A \mid K$ is a summand of $K$ then $A$ is semisimple and we have an injection $\text{Irr}(\mathcal{A} \mid A) \to \text{Irr}(\mathcal{A} \mid K)$.

**B.4.** We will denote by $A = \text{End}_{\mathcal{A}}(K)$ the endomorphism algebra of $K$, which is a finite dimensional $k$-algebra. We have a contravariant $k$-linear functor

$$\mathfrak{g}_K = \text{Hom}_{\mathcal{A}}(-, K) : \mathcal{A} \to A\text{-mod}$$

where $\text{Hom}_{\mathcal{A}}(A, K)$ is naturally a left $A$-module via left composition.

**Lemma B.5.** Recall our assumption that $K$ is semisimple. The algebra $A$ is semisimple and the functor $\mathfrak{g}_K$ defines a bijection $\text{Irr}(\mathcal{A} \mid K) \to \text{Irr}(A)$.

**Proof.** We have $A = \bigoplus_{j=1}^r A e_j$, where $e_j = m_jp_j \in A$ is an idempotent. For any $1 \leq j, k \leq r$ we have a $k$-linear isomorphism $\text{Hom}_{\mathcal{A}}(A_j, A_k) \to e_kAe_j$ defined by $f \mapsto m_jfp_k$ so $Ae_j$ is simple by Schur’s Lemma, see [12, 1.8.4] and [8, 3.18]. Thus $A$ is semisimple and $\mathfrak{g}_K(A_j)$ is simple because right multiplication by $p_j$ defines an isomorphism $\mathfrak{g}_K(A_j) \to Ae_j$ of $A$-modules.

The resulting map on isomorphism classes is surjective because every simple $A$-module is a submodule of $A$. Moreover, this is injective by Schur’s Lemma because we have standard $k$-linear isomorphisms

$$\text{Hom}_{\mathcal{A}}(\mathfrak{g}_K(A_j), \mathfrak{g}_K(A_k)) \cong \text{Hom}_{\mathcal{A}}(Ae_j, Ae_k) \cong e_jAe_k \cong \text{Hom}_{\mathcal{A}}(A_k, A_j)$$

where the second isomorphism is given by $f \mapsto e_jf(e_j)e_k$. ■

**Lemma B.6.** For any summand $A \mid K$ and object $B \in \mathcal{A}$ we have a $k$-linear isomorphism

$$\mathfrak{g}_K : \text{Hom}_{\mathcal{A}}(B, A) \to \text{Hom}_{\mathcal{A}}(\mathfrak{g}_K(A), \mathfrak{g}_K(B)).$$

**Proof.** By Lemma B.3 we can assume $A = A_j$ for some $1 \leq j \leq r$. Now assume $\varphi \in \text{Hom}_{\mathcal{A}}(B, A_j)$ satisfies $\mathfrak{g}_K(\varphi) = 0$ then $m_j\varphi = 0$ but as $m_j$ is a monomorphism this implies $\varphi = 0$, hence the map is injective. Counting dimensions, exactly as in the proof of [15, 4.1.2], we get the map is surjective. ■

**B.7.** We now assume that $\mathcal{B}$ is another locally finite $k$-linear abelian category and $\mathcal{J} : \mathcal{A} \to \mathcal{B}$ is a $k$-linear functor. If $L := \mathcal{J}(K)$ and $\mathcal{B} := \text{End}_{\mathcal{A}}(L)$ then $\mathcal{J}$ defines a $k$-algebra homomorphism $\mathcal{J} : A \to B$. In particular, we may view $B$ as an $(A, A)$-bimodule.
by restricting through \( J \). With this we have a corresponding functor \( B \otimes_A: \text{\texttt{A-mod}} \to \text{\texttt{B-mod}} \) where \( B \otimes_A M \), for any \( A \)-module \( M \in \text{\texttt{A-mod}} \), is a left \( B \)-module in the usual way.

**Lemma B.8.** If \( A \mid K \) is a summand of \( K \) then we have a \( B \)-module isomorphism

\[
\phi : B \otimes_A \mathfrak{J}_K(A) \to \mathfrak{J}_L(J(A))
\]

satisfying \( \phi(b \otimes f) = b \mathfrak{J}(f) \).

**Proof.** Let \( m : A \to K \) and \( p : K \to A \) be morphisms such that \( pm = \text{Id}_A \) and \( u = mp \in A \) is an idempotent. As

\[ J(p)J(m) = J(pm) = J(\text{Id}_A) = \text{Id}_J(A) \]

we have \( J(A) \mid J(K) \) and \( J(u) \) is an idempotent. By the universal property of the tensor product we have a \( B \)-module homomorphism \( \phi : B \otimes_A Au \to B J(u) \) satisfying \( \phi(b \otimes a) = b \mathfrak{J}(a) \). Moreover, we have a \( B \)-module homomorphism \( \psi : B J(u) \to B \otimes_A Au \) satisfying \( \psi(x) = x \otimes u \). As \( x \mathfrak{J}(u) = x \) for any \( x \in B J(u) \) we clearly have \( \phi \psi \) is the identity, hence \( \phi \) is an isomorphism. The statement now follows because right multiplication by \( p \) defines an isomorphism of \( A \)-modules \( \mathfrak{J}_K(A) \to Au \) and left multiplication by \( J(m) \) defines an isomorphism of \( B \)-modules \( B J(u) \to \mathfrak{J}_L(J(A)) \).

**Corollary B.9.** If \( L = \mathfrak{J}(K) \) is semisimple and \( B \mid L \) then for any \( A \mid K \) we have a \( k \)-linear isomorphism

\[
\text{Hom}_B(J(A), B) \to \text{Hom}_B(\mathfrak{J}_L(B), B \otimes_A \mathfrak{J}_K(A)).
\]

**Proof.** As \( L \) is semisimple we have a \( k \)-linear isomorphism

\[
\text{Hom}_B(J(A), B) \cong \text{Hom}_B(\mathfrak{J}_L(B), \mathfrak{J}_L(J(A))).
\]

by Lemma B.6. The statement now follows from Lemma B.8.

**References**


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