The Root Datum of $\text{SO}_{2n+1}(\mathbb{K})$

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Let $\mathbb{K} = \overline{\mathbb{F}}_p$ for some odd prime $p > 0$. We define the special orthogonal group to be $\text{SO}_{2n+1}(\mathbb{K}) = \{X \in \text{Mat}_{2n+1}(\mathbb{K}) \mid X^T Q_{2n+1} X = Q_{2n+1} \text{ and } \det(X) = 1\}$, where we define the orthogonal matrix $Q_k$, (for any natural number $k > 0$), to be

$$Q_k = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \in \text{Mat}_k(\mathbb{K}).$$

Consider the algebraic group $\text{GL}_k(\mathbb{K})$ then we have a standard $BN$-pair given by the subgroup of all upper triangular matrices $B_k(\mathbb{K}) \leqslant \text{GL}_k(\mathbb{K})$ and the subgroup of all monomial matrices $N_k(\mathbb{K}) \leqslant \text{GL}_k(\mathbb{K})$. Also recall that we have a semidirect product decomposition of the Borel subgroup $B_k(\mathbb{K}) = U_k(\mathbb{K})T_k$ where $U_k(\mathbb{K})$ is the subgroup of uni-upper triangular matrices and $T_k$ is the maximal torus of diagonal matrices. By [Gec03, Summary 1.7.9] we have that $B = B_{2n+1}(\mathbb{K}) \cap \text{SO}_{2n+1}(\mathbb{K})$ and $N = N_{2n+1}(\mathbb{K}) \cap \text{SO}_{2n+1}(\mathbb{K})$ will form a $BN$-pair for $\text{SO}_{2n+1}(\mathbb{K})$. Furthermore we will have $B = UT$ where $T = B \cap N$ is a maximal torus of diagonal matrices and $U = U_{2n+1}(\mathbb{K}) \cap \text{SO}_{2n+1}(\mathbb{K})$.

We can express a matrix $X \in B_{2n+1}(\mathbb{K})$ as

$$X = \begin{pmatrix} A & v & B \\ 0 & x & w \\ 0 & 0 & C \end{pmatrix},$$

where $B \in \text{Mat}_n(\mathbb{K})$, $A, C \in B_n(\mathbb{K})$, $v$ is a column vector, $w$ a row vector and $x \in \mathbb{K}$. We now wish to find conditions such that $X$ lies in $B$. For this to be true we must have $X$ satisfies the defining equations of the special orthogonal group, namely $X^T Q_{2n+1} X = Q_{2n+1}$ and $\det(X) = 1$. Considering the first equation we have

$$X^T Q_{2n+1} X = \begin{pmatrix} A^T & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A^T \end{pmatrix} \begin{pmatrix} 0 & 0 & Q_n \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A & v & B \\ 0 & x & w \\ 0 & 0 & C \end{pmatrix}.$$

$$= \begin{pmatrix} 0 & 0 & A^T Q_n \\ 0 & x & v^T Q_n \\ C^T Q_n & w^T & B^T Q_n \end{pmatrix} \begin{pmatrix} A & v & B \\ 0 & x & w \\ 0 & 0 & C \end{pmatrix}.$$

$$= \begin{pmatrix} 0 & 0 & A^T Q_n C \\ 0 & x^2 & xw + v^T Q_n C \\ C^T Q_n A & C^T Q_n v + x w^T & C^T Q_n B + w^T w + B^T Q_n C \end{pmatrix}.$$
Therefore if $X^T Q_{2n+1} X = Q_{2n+1}$ we must have the following equations are satisfied

\[ A^T Q_n C = Q_n, \]  
\[ x^2 = 1, \]  
\[ xw + v^T Q_n C = 0, \]  
\[ C^T Q_n B + w^T w + B^T Q_n C = 0. \]

From eq. (2) we can deduce immediately that $x = \pm 1$ and from eq. (1) that $C = Q_n(A^{-1})^T Q_n$. Using eq. (3) we get

\[ v^T = -xwC^{-1}Q_n \Rightarrow v = -xQ_n(C^{-1})^T w^T, \]
\[ \Rightarrow v = -xQ_n(Q_nAQ_n)w^T, \]
\[ \Rightarrow v = -xAQ_nw^T. \]

Finally we can rewrite eq. (4) as

\[ C^T Q_n B + B^T Q_n C = -w^T w \Rightarrow (Q_nA^{-1})Q_n B + B^T Q_n(Q_n(A^{-1})^T Q_n) = -w^T w, \]
\[ \Rightarrow Q_nA^{-1}B + B^T(A^{-1})^T Q_n = -w^T w, \]
\[ \Rightarrow B = AQ_nS, \]

where $S + S^T = -w^T w$. Considering the equation $\det(X) = 1$ we have

\[ \det(X) = \det(A)x \det(C) = x \det(A) \det((A^{-1})^T) \det(Q_n)^2 = x \]

and so $x = 1$. By these calculations we have our Borel subgroup $B$ is of the form

\[ B = \left\{ \begin{pmatrix} A & v & AQ_nS \\ 0 & 1 & w \\ 0 & 0 & Q_n(A^{-1})^T Q_n \end{pmatrix} \middle| S \in \text{Mat}_n(\mathbb{K}) \text{ with } S + S^T = -w^T w, \quad v := -AQ_nw^T \text{ and } A \in B_n(\mathbb{K}) \right\}. \]

(5)

It’s clear from our calculation of the matrices in $B$ that we also have

\[ U = \left\{ \begin{pmatrix} A & v & AQ_nS \\ 0 & 1 & w \\ 0 & 0 & Q_n(A^{-1})^T Q_n \end{pmatrix} \middle| S \in \text{Mat}_n(\mathbb{K}) \text{ with } S + S^T = -w^T w, \quad v := -AQ_nw^T \text{ and } A \in U_n(\mathbb{K}) \right\}. \]

(6)

We also want to determine the subgroup $N$. Assume $X \in N_{2n+1}(\mathbb{K})$ then $X$ is a monomial matrix which can be expressed as

\[ X = \sum_{i=-n}^{n} x_{i,\sigma(i)} E_{i,\sigma(i)} \quad \Rightarrow \quad X^T = \sum_{i=-n}^{n} x_{i,\sigma(i)} E_{\sigma(i),i}, \]
for some \(0 \neq x_{i,\sigma(i)} \in \mathbb{K}\) and \(\sigma \in S_{2n+1}\). Note that we think of \(S_{2n+1}\) as permutations on the set \(\{-n, \ldots, -1, 0, 1, \ldots, n\}\) instead of the usual \(\{1, \ldots, 2n + 1\}\) and \(E_{i,j}\) is an elementary matrix such that we index the rows, from top to bottom, and the columns, from left to right, by \(-n, \ldots, -1, 0, 1, \ldots, n\). We can also express the matrix \(Q_{2n+1}\) as

\[
Q_{2n+1} = \sum_{i=-n}^{n} E_{i,-i}.
\]

Then for \(X\) to be an element of \(N\) we must have \(X^T Q_{2n+1} X = Q_{2n+1}\) and \(\det(X) = 1\). So, examining the first equation we have

\[
X^T Q_{2n+1} X = \left( \sum_{i=-n}^{n} x_{i,\sigma(i)} E_{\sigma(i),i} \right) \left( \sum_{j=-n}^{n} E_{j,-j} \right) \left( \sum_{k=-n}^{n} x_{k,\sigma(k)} E_{k,\sigma(k)} \right),
\]

\[
= \sum_{i,j,k=-n}^{n} x_{i,\sigma(i)} x_{k,\sigma(k)} E_{\sigma(i),i} E_{j,-j} E_{k,\sigma(k)}.
\]

We only get non-zero terms when \(i = j\) and \(-j = k \Rightarrow k = -i\)

\[
= \sum_{i=-n}^{n} x_{i,\sigma(i)} x_{-i,\sigma(-i)} E_{\sigma(i),\sigma(-i)}.
\]

We now compare this to \(Q_{2n+1}\). If \(\sigma\) does not send the pair \((i, -i)\) to a pair of the same form then we must have

\[
x_{i,\sigma(i)} x_{-i,\sigma(-i)} = 0 \Rightarrow x_{i,\sigma(i)} = 0 \text{ or } x_{-i,\sigma(-i)} = 0.
\]

However these entries of the matrix were chosen to be non-zero and so we must have

\[
\sigma(i) = j \iff \sigma(-i) = -j,
\]

for \(-n \leq i, j \leq n\). Note that this property clearly implies that \(\sigma(0) = 0\). We therefore define \(W \leq S_{2n+1}\) to be the subgroup of permutations which permute the unordered pairs \((1, -1), \ldots, (n, -n)\). Hence \(W\) is all permutations of the \(n\) pairs together with a collection of sign changes, so we will have \(W \cong \mathbb{Z}_2 \wr S_n\).

If \(\sigma \in W\) then we must have \(x_{-i,\sigma(-i)} = x_{i,\sigma(i)}^{-1}\). In particular \(x_{0,0} = x_{0,0}^{-1} \Rightarrow x_{0,0} = \pm 1\). Now examining \(\det(X) = 1\) we can see that

\[
\det(X) = \text{sgn}(\sigma) \prod_{i=-n}^{n} x_{i,\sigma(i)} = \text{sgn}(\sigma) x_{0,0} = \pm \text{sgn}(\sigma).
\]

So, if \(\text{sgn}(\sigma) = 1\) then \(x_{0,0} = 1\) and if \(\text{sgn}(\sigma) = -1\) then \(x_{0,0} = -1\). So, we have
\[
N = \left\{ \sum_{i=0}^{n} x_{i,\sigma(i)} E_{i,\sigma(i)} \bigg| \sigma \in W, x_{i,\sigma(i)} \in \mathbb{K} \text{ such that } x_{-i,\sigma(-i)} = x_{\sigma(i)}^{-1} \text{ for } i \neq 0 \right. \\
\quad \quad \text{and } x_{0,0} = \text{sgn}(\sigma) \right\}.
\]

Note that by taking the intersection of \( B \) and \( N \) we see \( T \) is the subgroup of all diagonal matrices which will be of the form

\[
\text{diag}(t_{-n}, \ldots, t_{-1}, 1, t_{-1}^{-1}, \ldots, t_{-n}^{-1}),
\]

for some \( t_1, \ldots, t_n \in \mathbb{K}^\times \).

**Roots and Root Subgroups**

Note for background information on how roots are constructed see [Car93, Section 1.9]. Now we have determined the \( BN \)-pair for \( \text{SO}_{2n+1}(\mathbb{K}) \) we can go about determining the root datum. We first want to find the minimal proper subgroups of \( U \) which are normalised by \( T \). These will be connected unipotent subgroups of dimension 1 and so are isomorphic to the additive group \( \mathbb{K}^+ \). Each such subgroup, called a *root subgroup*, will give rise to a positive root of \( \text{SO}_{2n+1}(\mathbb{K}) \).

Recall the format of a matrix in \( U \) specified in eq. (6). We will find the 1-dimensional subgroups of \( U \) by setting as many parts of \( U \) equal to zero and then checking for stability under conjugation by the torus. Consider \( X \in U \) such that \( A \) is the \( n \times n \) identity matrix and \( w \) is the zero row vector. Then our matrix \( X \) has the form

\[
X = \begin{pmatrix}
I_n & 0 & Q_n S \\
0 & 1 & 0 \\
0 & 0 & I_n
\end{pmatrix},
\]

where \( S \in \text{Mat}_n(\mathbb{K}) \) is such that \( S = -S^T \). Note that \( S = -S^T \) automatically implies that all the diagonal entries of \( S \) are zero. We fix a minimal number of non-zero entries in \( S \). Setting such entries equal to \( \pm 1 \) then we can express \( X \) as

\[
X = I_{2n+1} + E_{-j,i} - E_{-i,j},
\]

for some fixed \( 1 \leq i, j \leq n \) with \( i \neq j \).

Let \( t = \text{diag}(t_{-n}, \ldots, t_{-1}, 1, t_1, \ldots, t_n) \in T \) be a diagonal matrix such that \( t_r = t_r^{-1} \) for all \( 1 \leq r \leq n \). We wish to now calculate the conjugate of \( X \) by \( t \).

\[
tXt^{-1} = \left( \sum_{k=-n}^{n} t_k E_{k,k} \right) (I_{2n+1} + E_{-j,i} - E_{-i,j}) \left( \sum_{\ell=-n}^{n} t_{\ell}^{-1} E_{\ell,\ell} \right),
\]

\[
= I_{2n+1} + \sum_{k,\ell=-n}^{n} t_k t_{\ell}^{-1} E_{k,k} E_{-j,i} E_{\ell,\ell} - t_k t_{\ell}^{-1} E_{k,k} E_{-i,j} E_{\ell,\ell},
\]

\[
= I_{2n+1} + t_{-j} t_i^{-1} E_{-j,i} - t_{-i} t_j^{-1} E_{-i,j},
\]
\[ I_{2n+1} + t_{-i} t_{-j} (E_{-j,i} - E_{-i,j}). \]

Hence for each fixed \( 1 \leq i < j \leq n \) the subgroup

\[ X_\alpha = \{ I_{2n+1} + \kappa (E_{-j,i} - E_{-i,j}) \mid \kappa \in \mathbb{K}^+ \} \]

is a unique root subgroup of \( U \) which is normalised by the torus. Note that we impose the condition \( i < j \) because for \( j < i \) we get an identical list of subgroups. The corresponding root to \( X_\alpha \) is given by

\[ \alpha(t) = t_{-i} t_{-j}. \]

We will now try and construct another collection of minimal subgroups. We continue to consider \( X \in U \), as specified in eq. (6), and let \( A = I_n \). However we let \( w \) be a row vector such that it has a single non-zero entry in the \( i \)th position, for some fixed \( 1 \leq i \leq n \). Then our matrix will have the form

\[
X = \begin{pmatrix} I_n & v & Q_n S \\ 0 & 1 & w \\ 0 & 0 & I_n \end{pmatrix}.
\]

Now \( v = -Q_n w^T \), which means \( v \) is column vector with only one non-zero entry in the \(-i\)th position. We want to consider the possible entries in the matrix \( Q_n S \) but we must first consider the condition \( S + S^T = -w^T w \). We have \( w \) contains only one non-zero entry, say \( w_i \), in the \( i \)th position and hence the \( n \times n \) matrix \(-w^T w \) will have only one non-zero entry \(-w_i^2\) in the \((i, i)\)th position. So we can consider \( S \) to be the matrix such that all entries are zero, except the \((i, i)\)th entry which is \(-\frac{1}{2} w_i^2\). Therefore we can express \( X \) as

\[
X = I_{2n+1} + w_i (E_{0,i} - E_{-i,0}) - \frac{1}{2} w_i^2 E_{-i,i}.
\]

Let \( t = \text{diag}(t_{-n}, \ldots, t_{-1}, 1, t_1, \ldots, t_n) \in T \) be a diagonal matrix such that \( t_r = t_{-r}^{-1} \) for all \( 1 \leq r \leq n \). We wish to now calculate the conjugate of \( X \) by \( t \) but to simplify matters we will choose \( w_i = 1 \).

\[
t X t^{-1} = \left( \sum_{k=-n}^{n} t_k E_{k,k} \right) \left( I_{2n+1} + E_{0,i} - E_{-i,0} - \frac{1}{2} E_{-i,i} \right) \left( \sum_{\ell=-n}^{n} t_\ell^{-1} E_{\ell,\ell} \right),
\]

\[
= I_{2n+1} + \sum_{k,\ell=-n}^{n} t_k t_\ell^{-1} E_{k,k} E_{0,i} E_{\ell,\ell} - t_k t_\ell^{-1} E_{k,k} E_{-i,0} E_{\ell,\ell} - \frac{1}{2} t_k t_\ell^{-1} E_{k,k} E_{-i,i} E_{\ell,\ell},
\]

\[
= I_{2n+1} + t_i^{-1} E_{0,i} - t_i E_{-i,0} - \frac{1}{2} t_i t_i^{-1} E_{-i,i},
\]

\[
= I_{2n+1} + t_{-i} (E_{0,i} - E_{-i,0}) - \frac{1}{2} t_i^2 E_{-i,i}.
\]
Hence for each fixed $1 \leq i \leq n$ the subgroup

$$X_\beta = \left\{ I_{2n+1} + \kappa(E_{0,i} - E_{-i,0}) - \frac{1}{2}\kappa^2 E_{-i,i} \mid \kappa \in \mathbb{K}^+ \right\}$$

is a unique root subgroup of $U$ which is normalised by the torus. The corresponding root to $X_\beta$ is given by

$$\beta(t) = t_{-i}.$$

We aim to finally construct the last family of minimal subgroups. We continue to consider $X \in U$, as specified in eq. (6) but instead let $S$ be the zero matrix and $w$ the zero vector. Then our matrix will have the form

$$X = \begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & Q_n(A^{-1})^T Q_n \end{pmatrix}.$$

We can choose $A$ to have precisely one non-zero entry. Taking this entry to be 1 we can express $X$ as

$$X = I_{2n+1} + E_{-j,i} - E_{i,j},$$

for some fixed $1 \leq i < j \leq n$. Note that we require $i < j$ as $A$ is an $n \times n$ uni-upper triangular matrix.

Let $t = \text{diag}(t_{-n}, \ldots, t_{-1}, 1, t_1, \ldots, t_n) \in T$ be a diagonal matrix such that $t_r = t_{-r}^{-1}$ for all $1 \leq r \leq n$. We wish to now calculate the conjugate of $X$ by $t$.

$$tXt^{-1} = \left( \sum_{k=-n}^{n} t_k E_{k,k} \right) (I_{2n+1} + E_{-j,i} - E_{i,j}) \left( \sum_{\ell=-n}^{n} t_{\ell}^{-1} E_{\ell,\ell} \right),$$

$$= I_{2n+1} + \sum_{k,\ell=-n}^{n} t_k t_{\ell}^{-1} E_{k,k} E_{-j,-i} E_{\ell,\ell} - t_k t_{\ell}^{-1} E_{k,k} E_{i,j} E_{\ell,\ell},$$

$$= I_{2n+1} + t_{-j} t_{-i}^{-1} E_{-j,-i} - t_j t_i^{-1} E_{i,j},$$

$$= I_{2n+1} + t_{-j} t_i (E_{-j,-i} - E_{i,j}).$$

Hence for each fixed $1 \leq i < j \leq n$ the subgroup

$$X_\gamma = \{ I_{2n+1} + \kappa(E_{-j,-i} - E_{i,j}) \mid \kappa \in \mathbb{K}^+ \}$$

is a unique root subgroup of $U$ which is normalised by the torus. The corresponding root to $X_\gamma$ is given by

$$\gamma(t) = t_{-i} t_j.$$
A Better Description of the Roots

We wish to give a more uniform description of the roots that we have discovered in the previous section. We start by introducing so called co-ordinate maps on the torus. Let \( t = \text{diag}(t_{-n}, \ldots, t_{-1}, t_1, \ldots, t_n) \in T \) be a typical element of the maximal torus. Then we define algebraic group homomorphisms \( \varepsilon_i : T \to \mathbb{K}^\times \) by

\[
\varepsilon_i(t) = t_{-i},
\]

for all \( 1 \leq i \leq n \). Therefore we can describe the roots \( \alpha, \beta \) and \( \gamma \) by

\[
\alpha = \varepsilon_i + \varepsilon_j \quad \beta = \varepsilon_i \quad \gamma = \varepsilon_j - \varepsilon_i,
\]

where \((\varepsilon_i + \varepsilon_j)(t) = \varepsilon_i(t)\varepsilon_j(t)\) and \((\varepsilon_j - \varepsilon_i)(t) = \varepsilon_j(t)\varepsilon_i(t)^{-1}\). Hence we can describe the set of positive roots for \( SO_{2n+1}(\mathbb{K}) \) as the set

\[
\Phi^+ = \{\varepsilon_i, \varepsilon_j \pm \varepsilon_i \mid 1 \leq i \leq n \text{ and } i < j \leq n\}.
\]

Note also that a simple system of roots can be given by the set

\[
\Delta = \{\varepsilon_n, \varepsilon_{i+1} - \varepsilon_i \mid 1 \leq i < n\}.
\]

The Coroots

We know that to each root \( \alpha : T \to \mathbb{K}^\times \) there is a corresponding coroot \( \alpha^\vee : \mathbb{K}^\times \to T \) such that, for all \( \lambda \in \mathbb{K}^\times \), we have \((\alpha \circ \alpha^\vee)(\lambda) = \lambda^2\). We define maps \( \varepsilon_i^\vee : \mathbb{K}^\times \to T \) for all \( 1 \leq i \leq n \) by

\[
\varepsilon_i^\vee(\lambda) = \text{diag}(1, \ldots, 1, \lambda, 1, \ldots, 1),
\]

where \( \lambda \in \mathbb{K}^\times \) is in the \( i \)-th entry of the diagonal. Then it’s easy to check that the coroots are given by

\[
\alpha^\vee = \varepsilon_i^\vee + \varepsilon_j^\vee \quad \beta^\vee = 2\varepsilon_i^\vee \quad \gamma^\vee = \varepsilon_j^\vee - \varepsilon_i^\vee,
\]

where \((\varepsilon_i^\vee + \varepsilon_j^\vee)(\lambda) = \varepsilon_i^\vee(\lambda)\varepsilon_j^\vee(\lambda)\) and \((2\varepsilon_i^\vee)(\lambda) = \varepsilon_i^\vee(\lambda)^2\). Recall that the coroots, which we denote \( \Phi^\vee \), also form a root system and hence we can describe the positive coroots as the set

\[
\Phi^{\vee+} = \{2\varepsilon_i^\vee, \varepsilon_i^\vee \pm \varepsilon_j^\vee \mid 1 \leq j < i \leq n\}.
\]

References
