Structure of Root Data and Smooth Regular Embeddings

Jay Taylor
New Perspectives in Representation Theory of Finite Groups
October 17th 2017

University of Arizona
$SL_n(q)$
Representation Theory is "hard"

$\text{SL}_n(q)$
Representation Theory is "hard"

\[ \text{SL}_n(q) \]

Representation Theory is "easier"

\[ \text{GL}_n(q) \]
Representation Theory is "hard"

\[ \text{SL}_n(q) \triangleleft \text{GL}_n(q) \]

Representation Theory is "easier"
Representation Theory is "hard"

\[ G := \text{SL}_n(q) \triangleleft \text{GL}_n(q) =: \tilde{G} \]

Representation Theory is "easier"
Representation Theory is "hard"

\[ G := \text{SL}_n(q) \triangleleft \text{GL}_n(q) =: \tilde{G} \]

Representation Theory is "easier"

Clifford’s Theorem

- \( \tilde{G}/G \) is cyclic \( \implies \) \( \text{Res}_G^\tilde{G}(\tilde{\chi}) = \chi_1 + \cdots + \chi_r \)
Representation Theory is "hard"

\[ G := \text{SL}_n(q) \triangleleft \text{GL}_n(q) =: \tilde{G} \]

Representation Theory is "easier"

**Clifford’s Theorem**

- \( \tilde{G}/G \) is cyclic \( \Rightarrow \) \( \text{Res}^{\tilde{G}}_G(\tilde{\chi}) = \chi_1 + \cdots + \chi_r \)
- \( g \in G \) and \( \langle g \rangle_{\tilde{G}} \cap G = \langle g \rangle_G \) then
  \[
  \chi_i(g) = \frac{\tilde{\chi}(g)}{r}
  \]
• $G$ a connected reductive algebraic group over $F = \overline{F}_p$
• $G$ a connected reductive algebraic group over $\mathbb{F} = \overline{\mathbb{F}}_p$

• $F : G \to G$ a Steinberg endomorphism
  \[ \sim \quad G^F = \{ g \in G \mid F(g) = g \} \text{ a finite reductive group} \]
• $G$ a connected reductive algebraic group over $\mathbb{F} = \overline{\mathbb{F}}_p$

• $F : G \to G$ a Steinberg endomorphism

$\sim \quad G^F = \{g \in G \mid F(g) = g\}$ a finite reductive group

**Philosophical Observation**

The representation theory of $G^F$ is harder when $Z(G)$ is disconnected.
• $G$ a connected reductive algebraic group over $\mathbb{F} = \overline{\mathbb{F}}_p$

• $F : G \to G$ a Steinberg endomorphism

$\sim \quad G^F = \{g \in G \mid F(g) = g\}$ a finite reductive group

**Philosophical Observation**

The representation theory of $G^F$ is harder when $Z(G)$ is disconnected.

Deligne–Lusztig (’76):
• $G$ a connected reductive algebraic group over $\mathbb{F} = \overline{\mathbb{F}}_p$

• $F : G \to G$ a Steinberg endomorphism

  $\mapsto G^F = \{ g \in G \mid F(g) = g \}$ a finite reductive group

---

**Philosophical Observation**

The representation theory of $G^F$ is harder when $Z(G)$ is disconnected.

---

Deligne–Lusztig (’76):

• Should embed $G^F \triangleleft \tilde{G}^F$ as a normal subgroup such that $Z(\tilde{G})$ is connected.
- \( G \) a connected reductive algebraic group over \( \mathcal{F} = \overline{\mathbb{F}}_p \)
- \( F : G \to G \) a Steinberg endomorphism
  \[ \rightsquigarrow \quad G^F = \{ g \in G \mid F(g) = g \} \] a finite reductive group

**Philosophical Observation**

The representation theory of \( G^F \) is harder when \( Z(G) \) is disconnected.

Deligne–Lusztig (’76):

- Should embed \( G^F \triangleleft \tilde{G}^F \) as a normal subgroup such that \( Z(\tilde{G}) \) is connected.
- \( \mathcal{F}^\times \times \cdots \times \mathcal{F}^\times \cong T \leq G \) an \( F \)-stable maximal torus
  \[ G \times_{Z(G)} T = (G \times T)/\{(z, z^{-1}) \mid z \in Z(G)\} \]
**Example (G = SL_n(𝔽))**

- if $p \nmid n$ then $G \times_{Z(G)} T \cong GL_n(𝔽)$
- if $n = p^k$ then $G \times_{Z(G)} T \cong G \times T$
Example \((G = SL_n(\mathbb{F}))\)

- if \(p \nmid n\) then \(G \times_{Z(G)} T \cong GL_n(\mathbb{F})\)
- if \(n = p^k\) then \(G \times_{Z(G)} T \cong G \times T\)

Regular Embedding (Lusztig ’88)

A closed embedding \(\iota : G \hookrightarrow \tilde{G}\) is a regular embedding if:

- \(\tilde{G} = \iota(G)Z(\tilde{G})\) and \(Z(\tilde{G})\) is connected
- \(F : \tilde{G} \rightarrow \tilde{G}\) is a Steinberg endomorphism and \(\iota \circ F = F \circ \iota\).

We then have \(G^F \cong \iota(G)^F \triangleleft \tilde{G}^F\).
Smooth Regular Embeddings

- Root datum: $(G, T) \mapsto \mathcal{R}(G, T) = (X(T), \Phi, \tilde{X}(T), \tilde{\Phi})$
Smooth Regular Embeddings

- Root datum: $(G, T) \leftrightarrow R(G, T) = (X(T), \Phi, \tilde{X}(T), \tilde{\Phi})$
  - $X(T) = \text{Hom}(T, F^\times) \cong \mathbb{Z}^n$
  - $\tilde{X}(T) = \text{Hom}(F^\times, T) \cong \mathbb{Z}^n$

(Recall: $T \cong F^\times \times \cdots \times F^\times$.)

$Z(G) \hookrightarrow T \twoheadrightarrow X(T) \twoheadrightarrow X(Z(G))$

$Z(G) = \bigcap_{\alpha \in \Phi} \ker(\alpha) \twoheadrightarrow X(T)/Z\Phi \sim X(Z(G))$

Lemma $Z(G)$ is connected if and only if $X(T)/Z\Phi$ has no $p'$-torsion.

Definition $Z(G)$ is smooth if $X(T)/Z\Phi$ has no $p$-torsion.
Smooth Regular Embeddings

- Root datum: \((G, T) \leftrightarrow \mathcal{R}(G, T) = (X(T), \Phi, \tilde{X}(T), \tilde{\Phi})\)
  - \(X(T) = \text{Hom}(T, F^\times) \cong \mathbb{Z}^n\)
  - \(\tilde{X}(T) = \text{Hom}(F^\times, T) \cong \mathbb{Z}^n\)

  (Recall: \(T \cong F^\times \times \cdots \times F^\times\).)

- \(Z(G) \hookrightarrow T \leadsto X(T) \twoheadrightarrow X(Z(G))\)
Smooth Regular Embeddings

- Root datum: \((G, T) \leftrightarrow \mathcal{R}(G, T) = (X(T), \Phi, \check{X}(T), \check{\Phi})\)
  - \(X(T) = \text{Hom}(T, F^\times) \cong \mathbb{Z}^n\)
  - \(\check{X}(T) = \text{Hom}(F^\times, T) \cong \mathbb{Z}^n\)
  (Recall: \(T \cong F^\times \times \cdots \times F^\times\).)

- \(Z(G) \hookrightarrow T \xrightarrow{\sim} X(T) \twoheadrightarrow X(Z(G))\)

- \(Z(G) = \bigcap_{\alpha \in \Phi} \text{Ker}(\alpha) \xrightarrow{\sim} (X(T)/\mathbb{Z}\Phi)_{p'} \cong X(Z(G))\)
Smooth Regular Embeddings

- Root datum: \((G, T) \leftrightarrow R(G, T) = (X(T), \Phi, \tilde{X}(T), \tilde{\Phi})\)
  - \(X(T) = \text{Hom}(T, F^\times) \cong \mathbb{Z}^n\)
  - \(\tilde{X}(T) = \text{Hom}(F^\times, T) \cong \mathbb{Z}^n\)

  (Recall: \(T \cong F^\times \times \cdots \times F^\times\).

- \(Z(G) \hookrightarrow T \xrightarrow{\sim} X(T) \twoheadrightarrow X(Z(G))\)

- \(Z(G) = \bigcap_{\alpha \in \Phi} \text{Ker}(\alpha) \hookrightarrow (X(T)/\mathbb{Z}\Phi)_{p'} \cong X(Z(G))\)

**Lemma**

\(Z(G)\) is connected if and only if \(X(T)/\mathbb{Z}\Phi\) has no \(p'\)-torsion.
Root datum: \((G, T) \leftrightarrow R(G, T) = (X(T), \Phi, \tilde{X}(T), \tilde{\Phi})\)

- \(X(T) = \text{Hom}(T, F^\times) \cong \mathbb{Z}^n\)
- \(\tilde{X}(T) = \text{Hom}(F^\times, T) \cong \mathbb{Z}^n\)

(Recall: \(T \cong F^\times \times \cdots \times F^\times\).)

- \(Z(G) \hookrightarrow T \leadsto X(T) \twoheadrightarrow X(Z(G))\)
- \(Z(G) = \bigcap_{\alpha \in \Phi} \ker(\alpha) \leadsto (X(T)/Z\Phi)_{p'} \cong X(Z(G))\)

**Lemma**

\(Z(G)\) is connected if and only if \(X(T)/Z\Phi\) has no \(p'\)-torsion.

**Definition**

\(Z(G)\) is **smooth** if \(X(T)/Z\Phi\) has no \(p\)-torsion.
Smooth Regular Embeddings

Example

\[ G = \text{GL}_n(K), \quad T \text{ diagonal matrices, then } \]
\[ X(T) \text{ has a natural basis } \{ e_1, \ldots, e_n \}. \]

We have
\[ Z(\Phi) = \{ a_1 e_1 + \cdots + a_n e_n | a_1 + \cdots + a_n = 0 \} = \text{Span} Z(\{ e_1 - e_2, \ldots, e_n - e_1 - e_n \}). \]

And
\[ X(T)/Z(\Phi) \text{ has no torsion as } X(T) = Z(\Phi) \oplus Z(e_n). \]

Definition
A regular embedding \( \iota: G \hookrightarrow \tilde{G} \) is a smooth regular embedding if
\[ Z(\tilde{G}) \text{ is connected and smooth.} \]

Example:
\( \text{SL}_n(F) \hookrightarrow \text{GL}_n(F) \).
Example

\( G = \text{GL}_n(K), \ T \text{ diagonal matrices, then } X(T) \text{ has a natural basis } \{e_1, \ldots, e_n\} \).
Smooth Regular Embeddings

Example

$G = \text{GL}_n(\mathbb{K})$, $T$ diagonal matrices, then $X(T)$ has a natural basis $\{e_1, \ldots, e_n\}$. We have

$$\mathbb{Z}\Phi = \{a_1 e_1 + \cdots + a_n e_n \mid a_1 + \cdots + a_n = 0\}$$

$$= \text{Span}_\mathbb{Z}\{e_1 - e_2, \ldots, e_{n-1} - e_n\}$$

and $X(T)/\mathbb{Z}\Phi$ has no torsion as $X(T) = \mathbb{Z}\Phi \oplus \mathbb{Z}e_n$. 

6/12
Example
$G = \text{GL}_n(\mathbb{K})$, $T$ diagonal matrices, then $X(T)$ has a natural basis $\{e_1, \ldots, e_n\}$. We have

$$
\mathbb{Z}\Phi = \{a_1e_1 + \cdots + a_ne_n \mid a_1 + \cdots + a_n = 0\}
= \text{Span}_{\mathbb{Z}}\{e_1 - e_2, \ldots, e_{n-1} - e_n\}
$$

and $X(T)/\mathbb{Z}\Phi$ has no torsion as $X(T) = \mathbb{Z}\Phi \oplus \mathbb{Z}e_n$.

Definition
A regular embedding $\iota : G \hookrightarrow \tilde{G}$ is a smooth regular embedding if $Z(\tilde{G})$ is connected and smooth.
### Example

Let $G = GL_n(K)$, $T$ diagonal matrices, then $X(T)$ has a natural basis $\{e_1, \ldots, e_n\}$. We have

\[
\mathbb{Z}\Phi = \{a_1e_1 + \cdots + a_ne_n \mid a_1 + \cdots + a_n = 0\}
\]

\[
= \text{Span}_\mathbb{Z}\{e_1 - e_2, \ldots, e_{n-1} - e_n\}
\]

and $X(T)/\mathbb{Z}\Phi$ has no torsion as $X(T) = \mathbb{Z}\Phi \oplus \mathbb{Z}e_n$.

### Definition

A regular embedding $\iota: G \hookrightarrow \tilde{G}$ is a **smooth regular embedding** if $Z(\tilde{G})$ is connected and smooth.

Example: $SL_n(F) \hookrightarrow GL_n(F)$.
Remark

If we have a regular embedding \( \iota : G \rightarrow \tilde{G} \) then

\[
\iota : T \rightarrow \tilde{T} \quad \leftrightarrow \quad X(\tilde{T}) \rightarrow X(T)
\]
Remark

If we have a regular embedding \( \iota : G \hookrightarrow \tilde{G} \) then

\[
\iota : T \hookrightarrow \tilde{T} \quad \iff \quad X(\tilde{T}) \rightarrow X(T)
\]

- \( R = R(G, T) = (X, \Phi, \tilde{X}, \tilde{\Phi}) \).
Remark

If we have a regular embedding $\iota : G \hookrightarrow \tilde{G}$ then

$$\iota : T \hookrightarrow \tilde{T} \iff X(\tilde{T}) \twoheadrightarrow X(T)$$

- $\mathcal{R} = \mathcal{R}(G, T) = (X, \Phi, \tilde{X}, \tilde{\Phi})$.
- $A := X/\mathbb{Z}\Phi$ then we have a surjective map $f : X \twoheadrightarrow A$. 
Remark

If we have a regular embedding $\iota : \mathcal{G} \hookrightarrow \tilde{\mathcal{G}}$ then

$$\iota : \mathcal{T} \hookrightarrow \tilde{\mathcal{T}} \iff X(\tilde{\mathcal{T}}) \twoheadrightarrow X(\mathcal{T})$$

- $\mathcal{R} = \mathcal{R}(\mathcal{G}, \mathcal{T}) = (X, \Phi, \tilde{X}, \tilde{\Phi})$.
- $A := X/\mathbb{Z}\Phi$ then we have a surjective map $f : X \twoheadrightarrow A$.
- $\mathcal{T} = (T, \emptyset, \tilde{T}, \emptyset)$ and $h : T \rightarrow A$

$$X \oplus_{(A, f, h)} T = \{(x, t) \in X \oplus T \mid f(x) = h(t)\}.$$ 

and a surjective homomorphism

$$\phi : X \oplus_{(A, f, h)} T \twoheadrightarrow X.$$
• We have \( (X \oplus (A,f,h) T)/\mathbb{Z}\Phi \cong T \) has no torsion.
• We have \( (X \oplus (A, f, h) \ T)/\mathbb{Z}\Phi \cong T \) has no torsion.
• \( X \oplus (A, f, h) \ T \xrightarrow{\sim} \mathcal{R} \oplus (A, f, h) \mathcal{T} = \mathcal{R}(G', T') \)
• We have \((X \oplus (A,f,h) T)/\mathbb{Z}\Phi \cong T\) has no torsion.
• \(X \oplus (A,f,h) T \xrightarrow{\sim} \mathcal{R} \oplus (A,f,h) T = \mathcal{R}(G', T')\)

**Lemma (T.)**

There exists a smooth regular embedding \(G \hookrightarrow \tilde{G}\).
• We have $(X \oplus_{(A,f,h)} T)/\mathbb{Z}\Phi \cong T$ has no torsion.
• $X \oplus_{(A,f,h)} T \sim R \oplus_{(A,f,h)} \mathcal{I} = \mathcal{R}(G', T')$

<table>
<thead>
<tr>
<th>Lemma (T.)</th>
</tr>
</thead>
</table>
| There exists a smooth regular embedding $G \hookrightarrow \tilde{G}$.

<table>
<thead>
<tr>
<th>Proposition (T.)</th>
</tr>
</thead>
</table>
| Assume $G_n$ is one of $\text{SL}_{n+1}(\mathbb{F})$, $\text{Sp}_{2n}(\mathbb{F})$, $\text{Spin}_{2n+1}(\mathbb{F})$, or $\text{Spin}_{2n}(\mathbb{F})$.


• We have \( (X \oplus (A, f, h) T)/\mathbb{Z}\Phi \cong T \) has no torsion.

• \( X \oplus (A, f, h) T \xrightarrow{\sim} \mathcal{R} \oplus (A, f, h) \mathcal{I} = \mathcal{R}(G', T') \)

**Lemma (T.)**

There exists a smooth regular embedding \( G \hookrightarrow \tilde{G} \).

**Proposition (T.)**

Assume \( G_n \) is one of

\( \text{SL}_{n+1}(\mathbb{F}), \text{Sp}_{2n}(\mathbb{F}), \text{Spin}_{2n+1}(\mathbb{F}), \text{or Spin}_{2n}(\mathbb{F}) \).

There exists a smooth regular embedding \( G_n \hookrightarrow \tilde{G}_n \) such that each Levi subgroup of \( \tilde{G}_n \) is isomorphic to

\[ \text{GL}_{n_1}(\mathbb{F}) \times \cdots \times \text{GL}_{n_r}(\mathbb{F}) \times \tilde{G}_m \]

where \( n = n_1 + \cdots + n_r + m \).
\( \mathcal{R} \) := the isomorphism classes of root data.
Isomorphism Classes of Root Data

- \( \mathcal{R} := \) the isomorphism classes of root data.
- We partition \( \mathcal{R} \) into smaller subsets

\[
\mathcal{R} = \bigsqcup_{[\mathcal{R}, \mathcal{T}, K] / \sim} \mathcal{R}[\mathcal{R}, \mathcal{T}, K]
\]

where \( \mathcal{R} = (X, \Phi, \check{X}, \check{\Phi}) \) is semisimple, \( \mathcal{T} = (T, \emptyset, \check{T}, \emptyset) \) is a torus, and \( \Phi \subseteq K \subseteq X \) is a submodule.
Isomorphism Classes of Root Data

- $\mathcal{R} :=$ the isomorphism classes of root data.
- We partition $\mathcal{R}$ into smaller subsets

$$
\mathcal{R} = \bigsqcup_{[\mathcal{R}, \mathcal{T}, \mathcal{K}] / \sim} \mathcal{R}[\mathcal{R}, \mathcal{T}, \mathcal{K}]
$$

where $\mathcal{R} = (X, \Phi, \tilde{X}, \tilde{\Phi})$ is semisimple, $\mathcal{T} = (T, \emptyset, \tilde{T}, \emptyset)$ is a torus, and $\Phi \subseteq K \subseteq X$ is a submodule.

**Remark**

Recall that $G = G_{\text{der}} Z^\circ(G)$. Assume $\mathcal{R}(G) \in \mathcal{R}[\mathcal{R}, \mathcal{T}, \mathcal{K}]$ then

$$
\mathcal{R}(G_{\text{der}}) \cong \mathcal{R} \quad \mathcal{R}(Z^\circ(G)) \cong \mathcal{T} \quad X(G_{\text{der}} \cap Z^\circ(G)) \cong (X/K)_{p'}
$$
• $\mathcal{R} = (X, \Phi, \tilde{X}, \tilde{\Phi})$ and $\mathcal{T} = (T, \emptyset, \tilde{T}, \emptyset)$
Isomorphism Classes of Root Data

- $\mathcal{R} = (X, \Phi, \tilde{X}, \tilde{\Phi})$ and $\mathcal{T} = (T, \emptyset, \tilde{T}, \emptyset)$
- $A = X/K$ and $f : X \rightarrow A$ the natural projection
Isomorphism Classes of Root Data

- \( \mathcal{R} = (X, \Phi, \tilde{X}, \tilde{\Phi}) \) and \( \mathcal{T} = (T, \emptyset, \tilde{T}, \emptyset) \)
- \( A = X/K \) and \( f : X \to A \) the natural projection
- \( h : T \to A \) a surjective homomorphism

Theorem (T.)
The map \( \text{Aut}(A) \to \mathcal{R} \) defined by \( \psi \mapsto \mathcal{R} \oplus (A, f, \psi \circ h) \) induces a bijection \( \text{Aut}(\mathcal{R}, f) \to \mathcal{R} \) \( \mathcal{R} \setminus \text{Aut}(A) / \text{Aut}(\mathcal{T}, h) \to \mathcal{R} \left[ \mathcal{R}, T, K \right] \).

Corollary (T.)
If \( A \) has \( s \) invariant factors and \( \text{rk}(T) \geq s + 1 \) then \( \mathcal{R} \left[ \mathcal{R}, T, K \right] \) has cardinality 1.
Isomorphism Classes of Root Data

• $\mathcal{R} = (X, \Phi, \tilde{X}, \tilde{\Phi})$ and $\mathcal{T} = (T, \emptyset, \tilde{T}, \emptyset)$

• $A = X/K$ and $f : X \rightarrow A$ the natural projection

• $h : T \rightarrow A$ a surjective homomorphism

**Theorem (T.)**

The map $\text{Aut}(A) \rightarrow \mathcal{R}$ defined by $\psi \mapsto \mathcal{R} \oplus (A, f, \psi \circ h) \mathcal{T}$ induces a bijection

$$\text{Aut}(\mathcal{R}, f)(A) \setminus \text{Aut}(A) / \text{Aut}(\mathcal{T}, h)(A) \rightarrow \mathcal{R}[\mathcal{R}, \mathcal{T}, K].$$
Isomorphism Classes of Root Data

- \( \mathcal{R} = (X, \Phi, \tilde{X}, \tilde{\Phi}) \) and \( \mathcal{T} = (T, \emptyset, \tilde{T}, \emptyset) \)
- \( A = X/K \) and \( f : X \to A \) the natural projection
- \( h : T \to A \) a surjective homomorphism

**Theorem (T.)**

The map \( \text{Aut}(A) \to \mathcal{R} \) defined by \( \psi \mapsto \mathcal{R} \oplus_{(A,f,\psi \circ h)} \mathcal{T} \) induces a bijection

\[
\text{Aut}_{(\mathcal{R},f)}(A) \setminus \text{Aut}(A) / \text{Aut}_{(\mathcal{T},h)}(A) \to \mathcal{R}[\mathcal{R}, \mathcal{T}, K].
\]

**Corollary (T.)**

If \( A \) has \( s \) invariant factors and \( \text{rk}(T) \geq s + 1 \) then \( \mathcal{R}[\mathcal{R}, \mathcal{T}, K] \) has cardinality 1.
Example

Assume $G$ has a smooth connected centre and $G_{\text{der}} \cong SL_n(F)$. How many isomorphism classes of such groups are there?
Example

Assume $G$ has a smooth connected centre and $G_{\text{der}} \cong \text{SL}_n(F)$. How many isomorphism classes of such groups are there?

- $\dim(Z^0(G)) = 0 \leadsto \text{None}$,
Example

Assume $G$ has a smooth connected centre and $G_{\text{der}} \cong \text{SL}_n(\mathbb{F})$.

How many isomorphism classes of such groups are there?

- $\dim(Z^0(G)) = 0 \Rightarrow \text{None}$,

- $\dim(Z^0(G)) = 1 \Rightarrow \varphi(n)/2$, e.g., $\text{GL}_n(\mathbb{F})$, 

- $\dim(Z^0(G)) = 2 \Rightarrow \text{there's only one}$, e.g., $\text{GL}_n(\mathbb{F}) \times (\mathbb{F} \times \mathbb{F})^k$. 


### Example

Assume $G$ has a smooth connected centre and $G_{\text{der}} \cong SL_n(\mathbb{F})$. How many isomorphism classes of such groups are there?

- $\dim(Z^\circ (G)) = 0 \leadsto$ None,
- $\dim(Z^\circ (G)) = 1 \leadsto \varphi(n)/2$, e.g., $GL_n(\mathbb{F})$,
- $\dim(Z^\circ (G)) = 2 \leadsto$ there’s only one, e.g., $GL_n(\mathbb{F}) \times (\mathbb{F}^\times)^k$. 
• Reduce proving a property \( P \) for \( (G, F) \) to the case where \( G_{\text{der}} \) is simple and simply connected. This assumes \( F : G \rightarrow G \) is a Frobenius endomorphism.

• Get new proofs of Asai’s results and extend them to show that they are compatible with Steinberg endomorphisms.