Character Sheaves and GGGRs

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- $G$ a connected reductive algebraic group defined over $\overline{\mathbb{F}}_p$.
- $F : G \to G$ a Frobenius endomorphism defining an $\mathbb{F}_q$-rational structure $G^F = \{ g \in G \mid F(g) = g \}$.
- Fix a prime $\ell \neq p$ and an algebraic closure $\overline{\mathbb{Q}}_\ell$. Interested in \[ \text{Irr}(G^F) \subset \text{Cent}(G^F) = \{ f : G^F \to \overline{\mathbb{Q}}_\ell \mid f(xgx^{-1}) = f(x) \} \]

**Problem**

Given $g \in G^F$ and $\chi \in \text{Irr}(G^F)$ describe $\chi(g)$.

Two main cases to consider:
- $g \in G_{ss}^F = \{ x \in G^F \mid p \nmid o(x) \}$
- $g \in G_{uni}^F = \{ x \in G^F \mid o(x) = p^a \}$
For any $F$-stable maximal torus $\mathbf{T} \leq \mathbf{G}$ and $\theta \in \text{Irr}(\mathbf{T}^F)$ we have a virtual character

$$R^G_T(\theta) \in \mathbb{Z} \text{Irr}(\mathbf{G}^F).$$

**Theorem (Deligne–Lusztig, 1976)**

For any $\chi \in \text{Irr}(\mathbf{G}^F)$ and $s \in \mathbf{G}_{ss}^F$ we have

$$\chi(s) = \sum_{(\mathbf{T}, \theta)/\sim} \langle R^G_T(\theta), \chi \rangle R^G_T(\theta)(s)$$

and

$$R^G_T(\theta)(s) = \frac{1}{|C_G^\circ(s)^F|} \sum_{x \in \mathbf{G}^F \atop x^{-1}sx \in \mathbf{T}^F} \theta(x^{-1}sx).$$
\[ \mathcal{D}_G := \text{the bounded derived category of } \overline{\mathbb{Q}}_\ell\text{-constructible sheaves on } G \]

\[ \mathcal{M}_G := \text{the category of } \overline{\mathbb{Q}}_\ell\text{-perverse sheaves on } G \]

- Can think of an object \( A \in \mathcal{D}_G \) as a bounded “complex”

\[
\cdots \rightarrow A_{i-1} \rightarrow A_i \rightarrow A_{i+1} \rightarrow \cdots
\]

of \( \overline{\mathbb{Q}}_\ell \)-sheaves on \( G \) such that for each \( i \in \mathbb{Z} \) the cohomology sheaf \( \mathcal{H}^i(A) \) is constructible.

- In particular, for each \( x \in G \), the stalk \( \mathcal{H}^i_x(A) \) is a finite dimensional \( \overline{\mathbb{Q}}_\ell \)-vector space. Furthermore we have \( \mathcal{H}^i_x(A) \neq 0 \) for only finitely many \( i \in \mathbb{Z} \).

**Definition**

A **character sheaf** of \( G \) is a \( G \)-equivariant simple object in \( \mathcal{M}_G \). We denote by \( \hat{G} \) the set of character sheaves of \( G \).
The Frobenius endomorphism $F : G \to G$ induces a functor

$$F^* : \mathcal{D} G \to \mathcal{D} G$$

which preserves $\hat{G}$. We say $A \in \mathcal{D} G$ is $F$-stable if there exists an isomorphism

$$\phi_A : F^* A \to A \in \mathcal{D} G.$$

We denote by $\hat{G}^F \subseteq \hat{G}$ the subset of $F$-stable character sheaves.

**Definition**

Assume now that $A \in \hat{G}^F$. For each $x \in G^F$ and $i \in \mathbb{Z}$ we have

$$\mathcal{H}_x^i(F^* A) = \mathcal{H}_{F(x)}^i(A) = \mathcal{H}_x^i(A)$$

and $\phi_A$ induces an automorphism $\phi_A : \mathcal{H}_x^i(A) \to \mathcal{H}_x^i(A)$. We define the characteristic function of $A$ to be $\chi_{A,\phi_A} : G^F \to \overline{\mathbb{Q}}_\ell$ given by

$$\chi_{A,\phi_A}(g) = \sum_{i \in \mathbb{Z}} (-1)^i \text{Tr}(\phi_A, \mathcal{H}_g^i(A)).$$
Theorem (Lusztig, 1986, 2012)

There exists a family of isomorphisms \( \{ \phi_A : F^* A \to A \mid A \in \hat{G}^F \} \) (unique up to multiplication by roots of unity) such that

\[
\{ \chi_{A,\phi_A} \mid A \in \hat{G}^F \}
\]

is an orthonormal basis for \( \text{Cent}(G^F) \).

Definition

We say \( A \in \hat{G} \) is unipotently supported if \( \mathcal{H}_u^i(A) \neq 0 \) for some \( i \in \mathbb{Z} \) and \( u \in G_{\text{uni}} \).
Assume $P \leq G$ is a parabolic with Levi complement $L \leq P$. Lusztig has defined a map

$$A_0 \in \hat{L} \leadsto \text{ind}^G_{L \subseteq P}(A_0) \in \mathcal{M}_G$$

called induction. The complex $\text{ind}^G_{L \subseteq P}(A_0)$ satisfies the following properties:

- $\text{ind}^G_{L \subseteq P}(A_0) = A_0$ if $L = P = G$.
- $\text{ind}^G_{L \subseteq P}(A_0)$ is semisimple and all indecomposable summands are character sheaves.
- for any $A \in \hat{G}$ there exists a Levi subgroup $L \leq P$ and a cuspidal character sheaf $A_0 \in \hat{L}$ such that $(A : \text{ind}^G_{L \subseteq P}(A_0)) \neq 0$. Furthermore the pair $(L, A_0)$ is unique up to $G$-conjugacy.

**Definition**

We say $A \in \hat{G}$ is cuspidal if $(A : \text{ind}^G_{L \subseteq P}(A_0)) \neq 0$ implies $L = P = G$. 
Theorem (Lusztig)

If $A_0 \in \hat{L}$ is cuspidal and unipotently supported then

$$A_0 = IC(O_0 Z^\circ(L), E_0 \boxtimes L)[\dim O_0 + \dim Z^\circ(L)]$$

where:

- $O_0 \subseteq G$ is a unipotent conjugacy class,
- $E_0$ is an $L$-equivariant cuspidal local system on $O_0$,
- $L$ is a tame local system on $Z^\circ(L)$.

Furthermore, the quotient group $W_G(L) = N_G(L)/L$ is a Weyl group and

$$\text{End}_{\mathcal{D}_G}(\text{ind}^G_{L \subseteq P}(A_0)) \cong \overline{Q}_\ell W_G(L, L)$$

In particular, we have a bijection

$$\{ A \in \hat{G} \mid (A : \text{ind}^G_L(A_0)) \neq 0 \} \longleftrightarrow \text{Irr}(W_G(L, L))$$
Denote by $\mathcal{N}_G$ the set of all pairs $\iota = (\mathcal{O}_\iota, \mathcal{E}_\iota)$ where:

- $\mathcal{O}_\iota \subset G$ is a unipotent class,
- $\mathcal{E}_\iota$ is a $G$-equivariant local system on $\mathcal{O}_\iota$.

**Theorem (Lusztig, 1984)**

Denote by $\nu \in \mathcal{N}_L$ the cuspidal pair $(\mathcal{O}_0, \mathcal{E}_0)$ and assume that $\mathcal{L} = \overline{\mathbb{Q}}_\ell$. Then there is a subset $\mathcal{I}(L, \nu) \subseteq \mathcal{N}_G$ and a natural bijection

$$\mathcal{I}(L, \nu) \to \{ A \in \hat{G} \mid (A : \text{ind}_L^G(A_0)) \neq 0 \}$$

$$\iota \mapsto K_\iota.$$

Hence also a bijection

$$\mathcal{I}(L, \nu) \to \text{Irr}(W_G(L))$$

$$\iota \mapsto E_\iota.$$
Let \( A \in \hat{G}^F \) be an \( F \)-stable summand of \( \text{ind}_L^G(A_0) \) then we can assume:

\[
F(L) = L \quad F(O_0) = O_0 \quad F^*E_0 \cong E_0 \quad F^*L \cong L.
\]

In particular we have:

- \( F \) induces an automorphism of \( W_G(L) \) and \( W_G(L, \mathcal{L}) \),
- If \( A \) is parameterised by \( E \in \text{Irr}(W_G(L, \mathcal{L})) \) then this is fixed by \( F \).

**Proposition**

Assume we fix an isomorphism \( \varphi_0 : F^*E_0 \to E_0 \) and an extension \( \tilde{E} \) of \( E \) to \( W_G(L, \mathcal{L}) \rtimes \langle F \rangle \) (similarly an extension \( \tilde{E}_l \) of \( E_l \)). Then this induces isomorphisms

\[
\phi_A : F^*A \to A \quad \phi_l : F^*K_l \to K_l
\]
Theorem (T., 2014)

\[
\chi_{A,\phi} \mid_{G_{uni}} = \sum_{\iota \in \mathcal{I}(L,\nu)^F} \langle \tilde{E}_\iota, \text{Ind}_{W_G(L,\mathcal{L})}.F(\tilde{E}) \rangle W_G(L).F \cdot \chi_{K_{\iota},\phi_{\iota}}
\]

Theorem (Lusztig, T.)

Let \( a_{\iota} = -\dim \mathcal{O}_{\iota} - \dim Z^\circ(L) \) then we have

\[
\chi_{K_{\iota},\phi_{\iota}} = (-1)^{a_{\iota}} q^{(\dim G + a_{\iota})/2} P_{\iota',\iota} Y_{\iota'}
\]

Theorem (Bonnafé, Shoji, Waldspurger)

Assume \( p \) is good for \( G \) and one of the following holds:

- \( Z(G) \) is connected and \( G/Z(G) \) is simple,
- \( G \) is \( \text{SL}_n(F_p) \), \( \text{Sp}_{2n}(F_p) \) or \( \text{SO}_n(F_p) \).

Then the functions \( Y_{\iota'} \) are explicitly computable.
Assume now that $p$ is good for $G$. By Kawanaka (1986) we have a map

$$u \in G^F_{uni} \mapsto \gamma_u \in \text{Cent}(G)$$

where $\gamma_u$ is the character of a generalised Gelfand–Graev representation. These satisfy the following properties:

- $\gamma_u$ is obtained by inducing a linear character from a $p$-subgroup of $G^F$,
- $\gamma_u = \gamma_v$ if $xux^{-1} = v$ for some $x \in G^F$,
- $\gamma_1$ is the regular character and $\gamma_u$ is a Gelfand–Graev character when $u$ is a regular element.

**Problem**

Describe the multiplicities $\langle \gamma_u, \chi \rangle$ for all $\chi \in \text{Irr}(G^F)$. 
Consider $G^F = \text{GL}_n(q)$ and $B$ the upper triangular matrices then

$$\text{Ind}_{B^F}^{G^F}(1_{B^F}) = \sum_{\rho \in \text{Irr}(S_n)} \rho(1)\chi_\rho$$

and

$$\mathcal{E}(G^F, 1) = \{\chi_\lambda \mid \lambda \vdash n\}$$

is the set of unipotent characters.

**Theorem (Kawanaka)**

$$\langle \Gamma_\mu, \chi_\lambda \rangle = \begin{cases} 
1 & \text{if } \lambda^* = \mu \\
0 & \text{if } \lambda^* \lhd \mu 
\end{cases}$$

**Example**

$\chi(n) = 1_{\text{GL}_n(q)}$ occurs in the regular representation with multiplicity 1 and in no other GGR.
If $p$ and $q$ are sufficiently large then Lusztig has given an explicit decomposition
\[ \gamma_u \mapsto \{ \chi_{K_\ell, \phi_\ell \mid \ell \in N_G^F} \} \]
and has conjectured an explicit decomposition
\[ \chi \in \text{Irr}(G^F) \mapsto \{ \chi_{A, \phi_A \mid A \in \widehat{G}^F} \} \]
If we solve this conjecture then the multiplicity $\langle \gamma_u, \chi \rangle$ can be reduced to the multiplicities
\[ \langle \chi_{A, \phi_A}, \chi_{K_\ell, \phi_\ell} \rangle \]
and these are given by our main theorem!