COMPUTING CHARACTER TABLES OF FINITE GROUPS

Jay Taylor
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Symmetry
Symmetry

Chemistry
Symmetry

Chemistry

Biology
Symmetry

Chemistry

Biology

Physics
Symmetry

Groups

Symmetry

Chemistry

Biology

Physics
Symmetry

Chemistry

Biology

Physics

Number Theory

Topology

Algebraic Geometry

Groups

Symmetry

Chemistry

Biology

Physics
DEFINITION
A group is a pair \((G, \star)\) with \(G\) a set and \(\star : G \times G \rightarrow G\) a binary operation such that:
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1. there exists an element \(e \in G\) such that \(x \star e = e \star x = x\) for all \(x \in G\)
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Examples

- $(\mathbb{Z}, +)$ with $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots \}$,
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Examples

- \((\mathbb{Z}, +)\) with \(\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}\),
- \((\mathbb{R}^\times, \times)\) with \(\mathbb{R}^\times = \mathbb{R} \setminus \{0\}\) where \(\mathbb{R}\) denote the real numbers.
DIHEDRAL GROUPS
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DIHEDRAL GROUPS
DIHEDRAL GROUPS
DIHEDRAL GROUPS
DIHEDRAL GROUPS
DIHEDRAL GROUPS
DIHEDRAL GROUPS

1  2

3  4

1  2

1  2

1  2
DIHEDRAL GROUPS
DIHEDRAL GROUPS
DIHEDRAL GROUPS

4

1

3

2
DIHEDRAL GROUPS

4

1

3

2

90°
DIHEDRAL GROUPS
DIHEDRAL GROUPS

4
1

3
2

90°
DIHEDRAL GROUPS
DIHEDRAL GROUPS
DIHEDRAL GROUPS

2
3
4

90°
DIHEDRAL GROUPS
DIHEDRAL GROUPS

1

4

3

2

90°

1
DIHEDRAL GROUPS

180°
DIHEDRAL GROUPS
DIHEDRAL GROUPS

3

4

2

1

90°

180°
DIHEDRAL GROUPS

1 4
2 3

90° 180°
DIHEDRAL GROUPS
DIHEDRAL GROUPS

1 4

2 3

90° 180°
DIHEDRAL GROUPS
DIHEDRAL GROUPS

1

90°

180°
DIHEDRAL GROUPS

2  3
1  4

270°
DIHEDRAL GROUPS
DIHEDRAL GROUPS

1. 90°
2. 180°
3. 270°
DIHEDRAL GROUPS

a

b

180°

90°

270°
DIHEDRAL GROUPS

a

b

ab

180°

90°

270°
DIHEDRAL GROUPS

- **a**
- **b**
- **ab**
- **aba**

- $180^\circ$
- $270^\circ$
- $90^\circ$
DIHEDRAL GROUPS

a

b

ab

aba

abab

180°

90°

270°
DIHEDRAL GROUPS

- **a**
- **b**
- **ab**
- **aba**
- **abab**
- **ababa**
- **180°**
- **270°**
- **90°**
DIHEDRAL GROUPS

\[ \begin{align*}
&\text{a} & \text{b} & \text{ab} & \text{aba} \\
&180^\circ & & 90^\circ & \\
&\text{abab} & \text{ababa} & \text{ababab} & \\
\end{align*} \]
DIHEDRAL GROUPS

a

b

ab

aba

abab

ababa

ababab

e
DIHEDRAL GROUPS

\[ a^2 = e, \]
DIHEDRAL GROUPS

\[ a^2 = e, \quad b^2 = e, \]
DIHEDRAL GROUPS

\[ a^2 = e, \quad b^2 = e, \quad (ab)^4 = e \]
Dihedral Groups

\[ l_2(4) = \langle a, b \mid a^2 = e, \quad b^2 = e, \quad (ab)^4 = e \rangle \]
DIHEDRAL GROUPS
DIHEDRAL GROUPS

\[ a, b, ab, aba, (ab)^2, (ab)^3, (ab)^3a, (ab)^4, e \]
DIHEDRAL GROUPS

$I_2(5) = \langle a, b \mid a^2, b^2, (ab)^5 \rangle$
DIHEDRAL GROUPS

\[
l_2(m) = \langle a, b \mid a^2, b^2, (ab)^m \rangle
\]
MATRIX REPRESENTATION
MATRIX REPRESENTATION
MATRIX REPRESENTATION
MATRIX REPRESENTATION
MATRIX REPRESENTATION

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]
MATRIX REPRESENTATION

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

\(\alpha\)
MATRIX REPRESENTATION

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0 & 1 \\
1 & 0 \\
\end{bmatrix}
\]
MATRIX REPRESENTATION

\[ \begin{bmatrix}
    0 & 1 \\
    1 & 0
\end{bmatrix} \]

\( \alpha \)
MATRIX REPRESENTATION

\[
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}
\quad \text{a}
\]

\[
\begin{bmatrix}
1 & 0 \\
0 & -1 \\
\end{bmatrix}
\quad \text{b}
\]
MATRIX REPRESENTATION

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\quad \quad
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]
MATRIX REPRESENTATION

\[
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 \\
0 & -1 \\
\end{bmatrix}
\]

\(a\)

\(b\)
MATRIX REPRESENTATION

\[ a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \]
MATRIX REPRESENTATION

\[
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 \\
0 & -1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & -1 \\
1 & 0 \\
\end{bmatrix}
\]

\(a\) \(b\) \(ab\)
MATRIX REPRESENTATION

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \quad \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix} \quad \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} \quad \begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}
\]

\[
a \quad b \quad ab \quad aba
\]

\[
\begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix} \quad \begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix} \quad \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \quad \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

\[
abab \quad ababa \quad ababab \quad e
\]
REPRESENTATIONS
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• $V$ an $n$-dimensional $\mathbb{C}$-vector space.
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- $\text{GL}(V) \cong \text{GL}_n(\mathbb{C})$ the group of all invertible linear transformations $f : V \to V$. 
• \( V \) an \( n \)-dimensional \( \mathbb{C} \)-vector space.

• \( \text{GL}(V) \cong \text{GL}_n(\mathbb{C}) \) the group of all invertible linear transformations \( f : V \to V \).

• A representation of a group \((G, \star)\) is a map \( \rho : G \to \text{GL}(V) \) such that for all \( a, b \in G \) we have

\[
\rho(a \star b) = \rho(a) \circ \rho(b)
\]
CHARACTERS

• Let $\rho : G \rightarrow GL_n(\mathbb{C})$ be a representation of a group $(G, \star)$. The function $\chi_\rho : G \rightarrow \mathbb{C}$ defined by

$$\chi_\rho(a) = \text{Tr}(\rho(a))$$

is called the character of $\rho$. 
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<table>
<thead>
<tr>
<th></th>
<th>e</th>
<th>a</th>
<th>b</th>
<th>ab</th>
<th>aba</th>
<th>abab</th>
<th>ababa</th>
<th>ababab</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\begin{bmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 &amp; 1 \ 1 &amp; 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 1 &amp; 0 \ 0 &amp; -1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 &amp; -1 \ 1 &amp; 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -1 &amp; 0 \ 0 &amp; 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -1 &amp; 0 \ 0 &amp; -1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 &amp; -1 \ -1 &amp; 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 &amp; 1 \ -1 &amp; 0 \end{bmatrix}$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
CONJUGACY
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• For any matrices $A, B \in \text{GL}_n(\mathbb{C})$ recall that

$$\text{Tr}(ABA^{-1}) = \text{Tr}(B)$$
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  $$\text{Tr}(ABA^{-1}) = \text{Tr}(B)$$

• Hence for any two elements $a, b \in G$ and any character $\chi : G \to \mathbb{C}$ we have
  
  $$\chi(a \star b \star a^{-1}) = \chi(b)$$
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- Hence for any two elements $a, b \in G$ and any character $\chi : G \to \mathbb{C}$ we have
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- We say $a, b \in G$ are **conjugate** if there exists an element $x \in G$ such that
  \[ x \star a \star x^{-1} = b \]
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• We say $a, b \in G$ are conjugate if there exists an element $x \in G$ such that
  \[ x \star a \star x^{-1} = b \]

• This defines an equivalence relation on $G$. The resulting equivalence classes are called conjugacy classes.
CONJUGACY

\[
\begin{align*}
  &a & &ababa \\
  &\text{90°} & &\text{270°} \\
  &ab & &ababab \\
  &\text{180°} & &\text{e}
\end{align*}
\]
IRREDUCIBLE CHARACTERS
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• A representation $\rho : G \rightarrow \text{GL}(V)$ is irreducible if there is no proper subspace $W \subseteq V$ which is invariant under $G$. By this we mean that for all $g \in G$ we have $\rho(g)W \subseteq W$. 
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• A character with this property is also called irreducible.
IRREDUCIBLE CHARACTERS

a

b
IRREDUCIBLE CHARACTERS

\[
\frac{1}{8} (2^2 + 0 + 0 + 0 + 0 + (-2)^2 + 0 + 0) = 1
\]
Theorem

The number of distinct irreducible characters of a finite group is equal to the number of conjugacy classes.
CHARACTER TABLES

Theorem
The number of distinct irreducible characters of a finite group is equal to the number of conjugacy classes.

Let $g_1, \ldots, g_n \in G$ be representatives for the conjugacy classes and let $\chi_1, \ldots, \chi_n$ be the irreducible characters of $G$. The square matrix

$$(\chi_i(g_j))_{1 \leq i, j \leq n}$$

is called the character table of $G$. 
## CHARACTER TABLES

<table>
<thead>
<tr>
<th>$l_2(4)$</th>
<th>$e$</th>
<th>$a$</th>
<th>$b$</th>
<th>$ab$</th>
<th>$(ab)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>$-1$</td>
<td>1</td>
<td>$-1$</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>1</td>
<td>1</td>
<td>$-1$</td>
<td>$-1$</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>1</td>
<td>$-1$</td>
<td>$-1$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\psi$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-2$</td>
</tr>
</tbody>
</table>
**CHARACTER TABLES**

<table>
<thead>
<tr>
<th>$I_2(2m)$</th>
<th>$e$</th>
<th>$a$</th>
<th>$b$</th>
<th>$(ab)^r$</th>
<th>$(ab)^m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
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<td>-1</td>
<td>1</td>
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<td>$(-1)^m$</td>
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</tr>
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<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_j$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>$\varepsilon^{jr} + \varepsilon^{-jr}$</td>
<td>$2(-1)^j$</td>
</tr>
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<td>$2(-1)^j$</td>
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</table>

$1 \leq j, r \leq m - 1 \quad \varepsilon = e^{\pi i/m}$
SYMMETRIC GROUPS
SYMMETRIC GROUPS

- $S_n$ is the group of all bijective functions $f : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$. 
SYMMETRIC GROUPS

- $\mathfrak{S}_n$ is the group of all bijective functions $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

- We call $\mathfrak{S}_n$ the symmetric group on $n$ points.
SYMMETRIC GROUPS

• $\mathcal{S}_n$ is the group of all bijective functions $f : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$.

• We call $\mathcal{S}_n$ the symmetric group on $n$ points.

• $|\mathcal{S}_n| = n!$ which can be very large even for small $n$. For example

$$|\mathcal{S}_{20}| = 2432902008176640000$$
SYMMETRIC GROUPS
Example \((n = 3)\)
SYMMETRIC GROUPS

Example \((n = 3)\)

\[
\begin{align*}
1 & \rightarrow 1 \\
2 & \rightarrow 2 \\
3 & \rightarrow 3
\end{align*}
\]

\[
\begin{align*}
1 & \rightarrow 1 \\
2 & \rightarrow 2 \\
3 & \rightarrow 3
\end{align*}
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SYMMETRIC GROUPS

Example \((n = 3)\)

\[
\begin{align*}
1 & \rightarrow 1 & 1 & \rightarrow 1 & 1 & \rightarrow 1 & 1 & \rightarrow 1 & 1 & \rightarrow 1 & 1 & \rightarrow 1 \\
2 & \rightarrow 2 & 2 & \rightarrow 2 & 2 & \rightarrow 2 & 2 & \rightarrow 2 & 2 & \rightarrow 2 & 2 & \rightarrow 2 \\
3 & \rightarrow 3 & 3 & \rightarrow 3 & 3 & \rightarrow 3 & 3 & \rightarrow 3 & 3 & \rightarrow 3 & 3 & \rightarrow 3
\end{align*}
\]
SYMMETRIC GROUPS

Example \((n = 3)\)

\[
\begin{array}{ccc}
1 & \rightarrow & 1 \\
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\end{array}
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\begin{array}{ccc}
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2 & \rightarrow & 2 \\
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2 & \rightarrow & 2 \\
3 & \rightarrow & 3
\end{array}
\]

\[
\begin{array}{ccc}
1 & \rightarrow & 1 \\
2 & \rightarrow & 2 \\
3 & \rightarrow & 3
\end{array}
\]
A function $f \in S_n$ is called a cycle of length $k$ if there exists a subset $X = \{x_1, \ldots, x_k\} \subseteq \{1, \ldots, n\}$ such that $f(i) = i$ for any integer $i \not\in X$ and $f$ acts on the elements of $X$ in the following way:

- $f(x_k) = x_1$
- $f(x_1) = x_2$
- $f(x_2) = x_3$
- $f(x_3) = x_4$
- $f(x_4) = x_k$

And so on, with the sequence repeating until $f(x_k) = x_1$.
Lemma

Every element of $\mathfrak{S}_n$ is a product of disjoint cycles.
SYMMETRIC GROUPS

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SYMMETRIC GROUPS
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- A partition of \( n \) is a sequence \( \mu = (\mu_1, \ldots, \mu_k) \) of integers such that \( \mu_1 \geq \cdots \geq \mu_k \geq 1 \) and \( \mu_1 + \cdots + \mu_k = n \).
SYMMETRIC GROUPS

• A partition of $n$ is a sequence $\mu = (\mu_1, \ldots, \mu_k)$ of integers such that $\mu_1 \geq \cdots \geq \mu_k \geq 1$ and $\mu_1 + \cdots + \mu_k = n$.

• For example the partitions of 5 are

  $(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1)$
SYMMETRIC GROUPS

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  (5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1)

- Given $f \in \mathfrak{S}_n$ let $f_1 \circ \cdots \circ f_k$ be a decomposition of $f$ into a product of disjoint cycles. If $\mu_i$ denotes the length of the cycle $f_i$ then the sequence $\mu(f) = (\mu_1, \ldots, \mu_k)$ is a partition of $n$, after possibly reordering the entries. We call $\mu(f)$ the cycle type of $f$. 
Theorem

Two elements of the symmetric group are conjugate if and only if they have the same cycle type.
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$$|P(20)| = 627$$
HOOKS OF PARTITIONS

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Theorem (Murnaghan–Nakayama Formula)

Write \( f \in \mathfrak{S}_n \) as a product \( f_1 \circ \cdots \circ f_k \) of disjoint cycles. Assume \( f_k \) is a cycle of length \( m \) then the element

\[
g = f_1 \circ \cdots \circ f_{k-1}
\]

is contained in the symmetric group \( \mathfrak{S}_{n-m} \). For any partition \( \lambda \in \mathcal{P}(n) \) we have

\[
\chi^\lambda(f) = \sum_{h_{ij} = m} (-1)^{l_{ij}} \chi^{\lambda \setminus R_{ij}}(g)
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