THE EULER CHARACTERISTIC OF A LIE GROUP

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1. Examples of Lie Groups

The following is adapted from [2]. We begin with the basic definition and some core examples.

Definition. A Lie group is a smooth manifold $G$ equipped with a group structure so that the maps $\mu : G \times G \to G; (x, y) \mapsto xy$ and $\iota : G \to G; x \mapsto x^{-1}$ are smooth.

Example.

- The manifold $\mathbb{R}^n$ forms a group under addition and hence is a Lie group.
- We have that $\mathbb{C}^n$ is diffeomorphic to $\mathbb{R}^{2n}$ and also forms a group under addition, hence is a Lie group.
- $\mathbb{R}^\ast := \mathbb{R} \setminus \{0\}$ is a smooth manifold and equipped with the standard multiplication in $\mathbb{R}$ we have $\mathbb{R}^\ast$ is a Lie group.

Lemma. Let $G_1, G_2$ be Lie groups. We can equip the product manifold $G := G_1 \times G_2$ with the product group structure. Then $G$ is a Lie group.

Proof. The multiplication map $\mu : G \times G \to G$ is given by $\mu((x_1, x_2), (y_1, y_2)) = [\mu_1 \times \mu_2](x_1, y_1)(x_2, y_2)$. Hence $\mu = (\mu_1 \times \mu_2) \circ (I_{G_1} \times S \times I_{G_2})$, where $S : G_2 \times G_1 \to G_1 \times G_2$ is defined by, $S(x_2, y_1) = (y_1, x_2)$. Therefore $\mu$ is a composition of smooth maps and so smooth. The inversion map of $G$ is given by $\iota = (\iota_1, \iota_2)$ and so is smooth. □

Lemma. Let $G$ be a Lie group and let $H \subset G$ be both a subgroup and a smooth submanifold. Then $H$ is also a Lie group.

Proof. Let $\mu_H : G \times G \to G$ be the multiplication map of $G$. Then the multiplication map of $H$ is given by $\mu_H = \mu|_{H \times H}$. This map is smooth because $\mu_G$ is smooth and $H \times H$ is a submanifold of $G \times G$. Now $H$ is a subgroup and so $\mu_H$ maps into the smooth submanifold $H$ and so $\mu_H$ is a smooth map into $H$. Similarly for $\iota_H$. □

Definition. Let $G$ and $H$ be Lie groups.

- A Lie group homomorphism is a smooth map $\varphi : G \to H$ which is also a homomorphism of groups.
- A Lie group isomorphism is a bijective Lie group homomorphism, whose inverse is also a Lie group homomorphism.
- An automorphism is a Lie group isomorphism from $G$ to itself.
Example. Consider the Lie group homomorphism \( \varphi = (\varphi_1, \ldots, \varphi_n) : \mathbb{R}^n \to \mathbb{T}^n \), where \( \mathbb{T}^n \) is the standard topological torus, given by \( \varphi_j(x) = e^{2\pi i x_j} \). We can easily verify that \( \varphi \) is a local diffeomorphism. Its kernel equals \( \mathbb{Z}^n \). Therefore we must have that this map factors through an isomorphism of groups, say \( \overline{\varphi} : \mathbb{R}^n / \mathbb{Z}^n \to \mathbb{T}^n \). Via this isomorphism we can pass the manifold structure of \( \mathbb{T}^n \) to a manifold structure of \( \mathbb{R}^n / \mathbb{Z}^n \). Therefore \( \mathbb{R}^n / \mathbb{Z}^n \) is a Lie group and \( \overline{\varphi} \) is an isomorphism of Lie groups.

We now want to show that the general linear group is in fact a Lie group, this will enable us to see that lots of other familiar groups are Lie groups as well.

Example. Let \( V \) be a real linear space of finite dimension \( n \), then we write \( \text{End}(V) \) for the linear space of all endomorphisms of \( V \). We write \( \text{GL}(V) \) for the set of invertible endomorphisms of \( V \), i.e.

\[
\text{GL}(V) := \{ A \in \text{End}(V) \mid \det A \neq 0 \}.
\]

Now the map \( \det : \text{End}(V) \to \mathbb{R} \) is a continuous map and \( \mathbb{R} \setminus \{0\} \) is an open subset of \( \mathbb{R} \). Therefore \( \det^{-1}(\mathbb{R} \setminus \{0\}) \) is an open subset of \( \text{End}(V) \). Therefore \( \text{GL}(V) \) is a smooth manifold of dimension \( n \). We now want to show that the group operation and inversion maps are smooth.

Choose a basis for \( V \), say \( v_1, \ldots, v_n \). Let \( \text{mat} : \text{End}(V) \to M(n, \mathbb{R}) \) be the map that takes \( A \in \text{End}(V) \) to it’s corresponding matrix \( \text{mat}A = (a_{ij}) \) with respect to the chosen basis. Clearly \( \text{mat} \) is a linear isomorphism. We can think of \( M(n, \mathbb{R}) \cong \mathbb{R}^{n^2} \) and the functions \( X_{ij}(A) = a_{ij}, \) for \( 1 \leq i, j \leq n \), as coordinate functions for \( \text{End}(V) \). The restriction of these form a global chart for \( \text{GL}(V) \). So we have

\[
X_{ij}(\mu(AB)) = \sum_{\ell=1}^{n} X_{i\ell}(A)X_{\ell j}(B),
\]

for \( A, B \in \text{GL}(V) \) and so it is clear that \( \mu \) is smooth. In terms of the chart we have

\[
\det = \sum_{\sigma \in S_n} \text{sgn}(\sigma)X_{1\sigma(1)} \cdots X_{n\sigma(n)}.
\]

Clearly \( \det \) is a smooth nowhere vanishing function and so the map \( A \mapsto (\det A)^{-1} \) is a smooth function on \( \text{GL}(V) \). We know that \( A^{-1} = \frac{1}{\det A} \text{adj}(A) \) by Cramer’s rule and so \( \iota \) is a smooth function on \( \text{GL}(V) \). Hence \( \text{GL}(V) \) is a Lie group.

There is a basic idea of ‘homogeneity’ in a Lie group. Essentially this is the idea that we can determine information about the whole Lie group by looking at only one point. Specifically by examining a small open neighbourhood of the identity element we can extrapolate information about the whole group.

Let \( G \) be a Lie group. If \( x \in G \) then the left translation map \( l_x : G \to G \) is a smooth map given by \( y \mapsto \mu(x, y) \). The map \( l_x \) is a bijection with smooth inverse \( l_{x^{-1}} \) and so \( l_x \) is a diffeomorphism of \( G \) onto itself. Similarly the right translation map \( r_x : G \to G \) is a diffeomorphism of \( G \) onto \( G \). So, given two points \( a, b \in G \) we can see that \( l_{ab^{-1}} \) and...
Lemma. Let $G$ be a Lie group and $H$ a subgroup. Then the following are equivalent:

(a) $H$ is a submanifold of $G$ at the point $h$;
(b) $H$ is a submanifold of $G$.

Proof. Omitted.

Example. Let $V$ be a finite dimensional real linear space, of dimension $n$. Then the special linear group is defined to be

$$\text{SL}(V) := \{ A \in \text{GL}(V) \mid \det A = 1 \}.$$ 

Clearly $\text{SL}(V)$ is a subgroup of $\text{GL}(V)$ as it is the kernel of the map $\det$. We wish to show that $\text{SL}(V)$ is a submanifold of $\text{GL}(V)$. By the above Lemma it will be enough to do this at the identity.

$\text{GL}(V)$ is an open subset of $\text{End}(V)$ and so we can identify $T_I(G)$ with $\text{End}(V)$. The $\det : G \to \mathbb{R}$ function is smooth and so its tangent map is a linear map from $\text{End}(V)$ to $\mathbb{R}$. In fact this tangent map is the trace function, $\text{tr} : \text{End}(V) \to \mathbb{R}$. Clearly the trace is a surjective linear map and so $\det$ is a submersion at $I$, (i.e. its derivative is everywhere surjective). The submersion theorem then tells us that $\text{SL}(V) = \det^{-1}(\{I\})$ is a smooth embedded submanifold of $\text{GL}(V)$ at $I$.

Theorem. Let $G$ be a Lie group and let $H$ be a subgroup of $G$. Then the following are equivalent:

(a) $H$ is closed, with respect to the topology of the manifold.
(b) $H$ is a submanifold.

Proof. Omitted.

Corollary. Let $G$ be a Lie group. Then every closed subgroup of $G$ is a Lie group.

Proof. If $H$ is a closed subgroup of $G$ then it is a submanifold by the above theorem. By a previous lemma any subgroup which is also a submanifold is a Lie group.

Corollary. Let $\varphi : G \to H$ be a homomorphism of Lie groups. Then the kernel of $\varphi$ is a closed subgroup of $G$. In particular, $\ker \varphi$ is a Lie group.

Proof. We have $\ker \varphi$ is a subgroup of $G$. Now $\varphi$ is continuous and $\{e_H\}$ is a closed subset of $H$ and so $\varphi^{-1}(\{e_H\})$ is a closed subset of $G$. Therefore by the above Corollary $\ker \varphi$ is a Lie group.

This Corollary quickly shows us that $\text{SL}(V)$ is a Lie group as it is the kernel of $\det : \text{GL}(V) \to \mathbb{R}^*$.

Example. Let $V$ be a finite dimensional complex linear vector space, of dimension $n$. Choose a basis, say $v_1, \ldots, v_n$, for $V$. Then the matrix map $\text{mat} : \text{End}(V) \to M(n, \mathbb{C})$ is a complex linear isomorphism. It restricts to Lie group isomorphisms $\text{GL}(V) \to \text{GL}(n, \mathbb{C})$ and $\text{SL}(V) \to \text{SL}(n, \mathbb{C})$. 

$r_{a-b}$ are diffeomorphisms of $G$ sending $a$ to $b$. Therefore we can compare structures on $G$ at different points.
We want to state a Lemma that will allow us to show that all the classical groups are in fact Lie groups. Let $V, W$ be finite dimensional real linear vector spaces and $\beta : V \times V \to W$ be a real bilinear form. We can define an action of $GL(V)$ on $\beta$ by $g \cdot \beta(u, v) = \beta(g^{-1}u, g^{-1}v)$. Hence the stabilizer of this action

$$GL(V)_\beta = \{ g \in GL(V) \mid g \cdot \beta = \beta \}$$

is a subgroup of $GL(V)$. Similarly $SL(V)_\beta = SL(V) \cap GL(V)_\beta$ is a subgroup.

**Lemma.** The stabilizers $GL(V)_\beta$ and $SL(V)_\beta$ are closed subgroups of $GL(V)$. In particular they are Lie groups.

**Proof.** Omitted. □

**Example.** Let $V = \mathbb{R}^n$ and $\beta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the standard Euclidean inner product, i.e. $\beta(x, y) = \sum_{i=1}^{n} x_i y_i$. Then $GL(V)_\beta = O(n)$, the orthogonal group and $SL(V)_\beta = SO(n)$, is the special orthogonal group.

**Example.** Let $V = \mathbb{R}^{2n}$ and let $\beta : \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ be the standard symplectic form given by

$$\beta(x, y) = \sum_{i=1}^{n} x_i y_{n+i} - \sum_{i=1}^{n} x_{n+i} y_i.$$

Then $GL(V)_\beta$ is the symplectic group $Sp(n)$.

**Example.** Let $V$ be a finite dimensional complex linear vector space, equipped with a hermitian inner product $\beta$. Note this is not a complex bilinear form as it is skew-symmetric in the second component. However as a map $V \times V \to \mathbb{C}$ it is bilinear over $\mathbb{R}$ and in particular is continuous. Therefore we have $GL(V)_\beta = U(V)$, the unitary group, (i.e. the group that preserves all hermitian inner products), is a closed subgroup of $GL(V)$ and hence a Lie group. Similarly the special unitary group $SU(V) := U(V) \cap SL(V)$ is a Lie group.

We have now given three very important examples of Lie groups, namely the orthogonal, symplectic and unitary groups. However, we defined these in terms of real bilinear forms but it is easy to see them more concretely as matrix groups. We use the previous results we have stated about $GL(V)$ for $GL(n, \mathbb{C})$, (or $GL(n, \mathbb{R})$ depending), by the isomorphism we specified previously.

**Example.** Let $\beta = \langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^n$. Then for any $A \in M(n, \mathbb{R})$ we have $\langle Ax, y \rangle = \langle x, A^t y \rangle$. Let $g \in GL(n, \mathbb{R})$ and $x, y \in \mathbb{R}^n$ then

$$g^{-1} \cdot \beta(x, y) = \langle gx, gy \rangle = \langle g^t gx, y \rangle.$$

Now we defined the orthogonal group $O(n)$ to be $GL(n, \mathbb{R})_\beta$ and so it is clear to see

$$O(n) = \{ g \in GL(n, \mathbb{R}) \mid g^t g = I \}.$$
Example. Let $\beta = \langle \cdot, \cdot \rangle$ be the standard hermitian inner product on $\mathbb{C}^n$, i.e. $\langle x, y \rangle = \sum_{i=1}^n x_i \overline{y}_i$. Then $\langle Ax, y \rangle = \langle x, A^\dagger y \rangle$ for $A \in M(n, \mathbb{C})$ where $A^\dagger$ denotes complex conjugate transpose. Now again we have for $g \in \text{GL}(n, \mathbb{C})$ and $x, y \in \mathbb{C}^n$ that

$$g^{-1} \cdot \beta(x, y) = \langle gx, gy \rangle = \langle g^\dagger gx, y \rangle.$$  

Now we defined the unitary group $U(n)$ to be $\text{GL}(n, \mathbb{C})_\beta$ and so it is clear to see

$$U(n) = \{ g \in \text{GL}(n, \mathbb{C}) \mid g^\dagger g = I \}.$$  

Example. Let $\beta$ be the standard symplectic form on $\mathbb{R}^{2n}$ as defined previously. Let $Q \in M(2n, \mathbb{R})$ be defined by

$$Q = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$  

Let $\langle \cdot, \cdot \rangle$ be the standard Euclidean inner product on $\mathbb{R}^{2n}$. Then for all $x, y \in \mathbb{R}^{2n}$ we have $\beta(x, y) = \langle x, Qy \rangle$. Let $g \in \text{GL}(n, \mathbb{R})$ then

$$g^{-1} \cdot \beta(x, y) = \langle gx, Qgy \rangle = \langle x, g^t Qgy \rangle.$$  

Now we defined the symplectic group $\text{Sp}(n)$ to be $\text{GL}(n, \mathbb{R})_\beta$ and so it is clear to see

$$\text{Sp}(n) = \{ g \in \text{GL}(n, \mathbb{R}) \mid g^t Qg = Q \}.$$  

2. The Euler Characteristic

Throughout this section any Lie group $G$ will be compact and connected.

We note that the following section is adapted from [1].

Definition. Let $G$ be a Lie group then we say $T \subset G$ is a maximal torus if $T$ is a maximal, (with respect to inclusion), abelian subgroup of $G$.

Let $T$ be a maximal torus of a Lie group $G$. Then we will find the Euler characteristic of $G/T$ as a consequence of showing a deeper result that any element of $G$ is contained in a conjugate of $T$. Before pursuing this theorem we first state, without proof, the following important theorem.

Theorem (Lefschetz’s Fixed Point Theorem). Let $X$ be a compact topological space such that there exists a simplicial complex $K$ that is homeomorphic to $X$, (i.e. $X$ is triangulable). Let $f: X \to X$ be a continuous map then we have this map induces a map on homology groups, which we denote $f^*: H^q(X; \mathbb{Q}) \to H^q(X; \mathbb{Q})$. We define the Lefschetz Number of $f$ as

$$\Lambda_f := \sum_{q \geq 0} (-1)^q \text{tr } f^*.$$  

Then if $\Lambda_f \neq 0$ then we must have $f$ has a fixed point, i.e. there exists an $x \in X$ such that $f(x) = x$. 

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Remark. We note that the Lefschetz number depends only upon the induced map on homology groups of $X$. Therefore this means that this theorem will hold for any map homotopic to $f$. We also note that the converse of this theorem is not true. We may have that $f$ has fixed points but the Lefschetz number is 0.

**Definition.** Let $G$ be a Lie group and $H$ a subgroup generated by $g$. Now $g$ is a generator of $G$ if $\overline{H} = G$.

**Proposition.** The standard topological torus $\mathbb{T}^k$ has a generator. In fact the generators are dense in $\mathbb{T}^k$.

**Proof.** Let $U_1, U_2, \ldots$ be a countable base for the open sets of $\mathbb{T}^k$. Consider $\mathbb{T}^k$ as $\mathbb{R}^k/\mathbb{Z}^k$ then we have coordinates $(x_1, \ldots, x_k)$ on $\mathbb{T}^k$. Then we define a cube to be a set \( \{ x \in \mathbb{T}^k \mid |x_i - \zeta_i| \leq \varepsilon \} \) for some fixed point $\zeta$ and real $\varepsilon > 0$.

Let $C_0$ be any cube. Suppose, inductively, that we have defined $C_0 \supset C_1 \supset \cdots \supset C_{m-1}$ and that $C_{m-1}$ has side $2\varepsilon$. Then there is an integer $N$ such that $N \cdot 2\varepsilon > 1$ and $N \cdot C_{m-1} = \mathbb{T}^k$. We can find $C_m \subset C_{m-1}$ such that $N \cdot C_m \subset U_m$. Let $g \in \cap_mC_m$. Then $gN \subset U_m$ and so $g$ is a generator of $\mathbb{T}^k$. \(\square\)

Having gathered enough small facts we can now go on to prove our main theorem.

**Theorem.** Let $G$ be a Lie group and $\mathbb{T} \subset G$ be a maximal torus of $G$. Then for any $g \in G$ there exists $h \in G$, $t \in \mathbb{T}$ such that $g = hth^{-1}$.

**Proof.** Consider the left coset space $G/\mathbb{T}$ and let $f : G/\mathbb{T} \to G/\mathbb{T}$ be defined by left multiplication with $g$, i.e. $f(x\mathbb{T}) = gx\mathbb{T}$. Then a fixed point of $f$ is a coset $x\mathbb{T}$ such that $gx\mathbb{T} = x\mathbb{T} \Rightarrow x^{-1}gx\mathbb{T} = \mathbb{T} \Rightarrow x^{-1}gx \in \mathbb{T}$, in other words $g \in x\mathbb{T}x^{-1}$. Hence to prove our theorem we only need to show that the map $f$ has a fixed point. To do this we will use Lefschetz fixed point theorem.

We recall from the statement of Lefschetz fixed point theorem that the Lefschetz number of $f$ is defined to be

$$\Lambda_f = \sum_{q=1}^n (-1)^q \text{tr} f^q.$$ 

Note that $\Lambda_f$ is precisely the definition of the Euler characteristic as $\text{tr} f^q$ is the rank of the homology group. If $f$ has only finitely many fixed points then $\Lambda_f$ counts the number of fixed points with multiplicity, which is defined as follows. Let $X$ be a smooth manifold and $x$ a fixed point of a differentiable map $f : X \to X$. Then consider $1 - f^* : T_xX \to T_xX$, (where $T_xX$ denotes the tangent space of $X$ at $x$). If $\det (1 - f^*) > 0$, then $f$ has multiplicity $+1$ at $x$ and if $\det (1 - f^*) < 0$ then $f$ has multiplicity $-1$ at $x$.

Now what we want to do is compute $\Lambda_f$. By the remark after the statement of the theorem it suffices to do this with any map $f_0$, which is homotopic to $f$. So we can replace $g$ in the definition of $f$ with any other $g_0 \in G$ as $G$ is path-connected. Take $g_0$ to be a generator of $\mathbb{T}$ and then define $f_0 : G/\mathbb{T} \to G/\mathbb{T}$ by $f_0(x\mathbb{T}) = g_0x\mathbb{T}$. Now the fixed points of $f_0$ are the cosets $n\mathbb{T}$ where $n \in N_G(\mathbb{T})$, (the normaliser of the torus in $G$). To see this let $n \in N_G(\mathbb{T})$ then
\[f_0(nT) = g_0nT = n(n^{-1}g_0n)T = nT,\]

as \(g_0 \in T, n \in N_G(T)\) then \(n^{-1}g_0nT = T\) and so \(nT\) is a fixed point of \(f_0\). Conversely, assume \(xT\) is a fixed point of \(f_0\) then \(gxxT = xT \Rightarrow x^{-1}g_0xT = T \Rightarrow x^{-1}g_0x \in T\) and so \(x \in N_G(T)\) because \(g_0\) generates \(T\).

We wish to examine \(N_G(T)\). Now \(N_G(T)\) is a closed subgroup of \(G\) and so is a Lie group. The irreducible component of \(N_G(T)\), written \(N_G(T)_1\), containing the identity is open and so has only a finite number of cosets. We claim that \(\text{det}(1-f_t)\) is trivial and so we have this subset only contains the trivial action.

Before we carry on we need a small result from Representation theory. Note this is stated without proof.

**Definition.** Let \(F\) be a field, (i.e. \(\mathbb{R}\) or \(\mathbb{C}\)) and let \(G\) be a topological group. Then an \(FG\)-\textbf{space} is a finite dimensional vector space \(V\) over \(F\) such that there exists a continuous homomorphism \(\theta : G \to \text{Aut} V\). This may also be called a \textbf{representation} of \(G\).

**Proposition.** \(\text{If } T \text{ is a torus of } G \text{ then the torus acts on } T_1G \text{ by the adjoint action } G \to \text{Aut } T_1G.\)

**Note.** We define the adjoint action in the following way. Consider the map \(\varphi : G \to \text{Aut}(G)\) defined as \(g \mapsto \varphi_g\) where \(\varphi_g\) is the standard inner automorphism of \(G\), such that \(\varphi_g(h) = ghg^{-1}\) for all \(h \in G\). Then the derivative at the identity of the map \(\varphi_g\) is an automorphism of the associated Lie algebra, we denote this map by \(\text{ad}_g : g \to \mathfrak{g}\). Then the adjoint action is the map \(\text{ad} : G \to \text{Aut}(T_1G)\) defined as \(g \mapsto \text{ad}_g\).
We choose a positive definite symmetric form on $T_1 G$ which is invariant under $G$ and so under $T$ also. Then $T_1 G$ decomposes into a direct sum of orthogonal irreducible $T$-spaces, say $V_0 \oplus \bigoplus_{i=1}^N V_i$. The vector space $V_0$ is of dimension 1 and the other $V_i$ are of dimension 2. Now $T$ acts trivially on $V_0$ and for the dimension 2 vector spaces we can choose an orthonormal basis of each $V_i$ and represent $T$ by $T \to SO(2, \mathbb{R})$. Therefore $T$ acts on the $V_i$ as

$$
\begin{pmatrix}
\cos 2\pi \theta_i(t) & -\sin 2\pi \theta_i(t) \\
\sin 2\pi \theta_i(t) & \cos 2\pi \theta_i(t)
\end{pmatrix}.
$$

Where $\theta_i : T \to \mathbb{R}/\mathbb{Z}$ is a non-zero map which never takes integer values. This prevents the action of $T$ on the $V_i$ from being trivial.

Now we have this proposition we can state that $1 - f_0'$, with respect to the chosen basis, has matrix form

$$
\begin{pmatrix}
A_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & A_m
\end{pmatrix}
$$

where $A_i := \begin{pmatrix}
1 - \cos 2\pi \theta_i(g_0) & \sin 2\pi \theta_i(g_0) \\
-\sin 2\pi \theta_i(g_0) & 1 - \cos 2\pi \theta_i(g_0)
\end{pmatrix}$.

Now we can see that

$$
\det A_i = \begin{vmatrix}
1 - \cos 2\pi \theta_i(g_0) & \sin 2\pi \theta_i(g_0) \\
-\sin 2\pi \theta_i(g_0) & 1 - \cos 2\pi \theta_i(g_0)
\end{vmatrix} = 2(1 - \cos 2\pi \theta_i(g_0))
$$

and $\det(1 - f_0') = \prod_{i=1}^m \det A_i$. Now $\det A_i$ is greater than 0 unless $\theta_i(g_0)$ is integer valued but the proposition tells us this never happens. Therefore the multiplicity of $T$ is +1 and hence each coset has multiplicity +1. So the Lefschetz number is $\Lambda_f = |N_G(T)/T| > 0$. Therefore $f$ has at least one fixed point, which proves the theorem. \hfill $\square$

**Remark.** In proving this theorem we have seen that the Euler characteristic of $G/T$ is in fact just $|N_G(T)/T|$ but why is this special? Well for any Lie group $G$ we have for a given maximal torus $T \subset G$ that $N_G(T)/T$ is in fact a Coxeter group. This is a group that has the following special form, given by generators and relations,

$$
\langle s_1, \ldots, s_t \mid s_i^2 = 1 \text{ and } (s_i s_j)^{m_{ij}} = 1 \text{ for some } m_{ij} \in \mathbb{N} \rangle.
$$

In fact it can be shown that $m_{ij}$ must be either 2, 3, 4 or 6. A standard example of a Coxeter group is the symmetric group $S_n$ which is generated by transpositions. You may be slightly concerned that this Coxeter group depends on the choice of maximal torus. However as a Corollary to the above theorem we have any two maximal tori of $G$ are conjugate. Therefore any construction depending upon a maximal torus is unique up to an inner automorphism of $G$.

The Coxeter group is one the most important components of a Lie group as it determines almost everything about the Lie group. It is essentially the symmetry group of the root system of the Lie group. In fact the Coxeter group of a Lie group will have finite order and
so infinite Lie groups are in fact controlled, primarily, by something that is finite. When we try to classify the Lie groups we can essentially reduce it to the problem of classifying all the finite Coxeter groups, which is a much simpler problem.

3. Examples of Euler Characteristic

(a) Consider the matrix group SU(2). Now any matrix $X \in SU(2)$ will have the form

$$X = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},$$

for some $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$. We can define an embedding $\varphi : \mathbb{C}^2 \to M(2, \mathbb{C})$, (considering $\mathbb{C}^2$ diffeomorphic to $\mathbb{R}^4$ and $M(2, \mathbb{C})$ diffeomorphic to $\mathbb{R}^8$), by

$$\varphi(\alpha, \beta) = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$ 

Now considering the restriction of $\varphi$ to $S^3$ we see that $\varphi$ is an embedding of $S^3$ into a compact submanifold of $\mathbb{R}^8$. However, we also have $\varphi(S^3) = SU(2)$. Therefore $S^3$ is diffeomorphic to SU(2) and hence $S^3$ is a Lie group. Now a maximal torus of SU(2) will be the subgroup $T$ of diagonal matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}.$$ 

The normaliser $N_G(T)$ will in fact be the matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}.$$ 

Hence the Coxeter group of SU(2) will have order 2. Therefore the Euler characteristic of SU(2)/T is 2. In particular this also tells us that the Euler characteristic of $S^3/S^1$ is 2. This makes sense as $S^3/S^1$ is diffeomorphic to $\mathbb{C}P^1$ which we know has Euler characteristic 2.

(b) In general SU($n$) will have Coxeter group $\mathfrak{S}_n$ and hence SU($n$)/T has Euler characteristic $n!$.

(c) The group SO($2n+1$) has associated Coxeter group $(\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$. A maximal torus of SO($2n+1$) is all diagonal matrices of the form

$$\text{diag}(d_1, \ldots, d_n, 1, d_n^{-1}, \ldots, d_1^{-1}).$$

Hence the Euler characteristic of SO($2n+1$)/T is the order of the Coxeter group which is $2^n n!$.

References