1. Definition and Examples

Any Lie Algebra is a vector space (with extra features). Revision: See Moodle.

**Definition:** Let $\mathbb{F}$ be any field (say $\mathbb{F} = \mathbb{C}$, which is algebraically closed, which means that any polynomial $a_0x^n + a_1x^{n-1} + \cdots + a_n$, with $a_0, a_1, \ldots, a_n \in \mathbb{C}$, has a root). Then a Lie algebra is a vector space $\mathfrak{g}$ with an operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$; $(X, Y) \mapsto [X, Y]$ called a Lie bracket, such that:

(a) **Antisymmetry:** $[Y, X] = -[X, Y]$ 
(b) **Bilinearity:** for all $a, b \in \mathbb{F}$ and for all $X, Y, Z \in \mathfrak{g}$
   (i) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
   (ii) $[X, aY + bZ] = a[X, Y] + b[X, Z]$.
(c) **The Jacobi Identity:** for all $X, Y, Z \in \mathfrak{g}$, $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

**Remark** (Comments on the Axioms).

1. Axiom (a) can be replaced by a (weaker) property $[X, X] = 0$ for all $X \in \mathfrak{g}$. Indeed, 

   (a) $\Rightarrow [X, X] = -[X, X]$ for $X = Y \Rightarrow [X, X] = 0$

   (provided characteristic $\mathbb{F} \neq 2$). Conversely, suppose we know $[X, X] = 0$ for all $X \in \mathfrak{g}$. Then

   $$[X + Y, X + Y] = 0 \Leftrightarrow [X, Y] + [Y, X] + [X, X] + [Y, Y] = 0$$

   $$\Leftrightarrow [X, Y] + [Y, X] = 0$$

2. In axiom (b) it is enough to assume either (i), or (ii), only. For example, let us show (i) $\Rightarrow$ (ii).

   $$[X, aY + bZ] = -[aY + bZ, X]$$ by (a)

   $$= -a[Y, X] - b[Z, X]$$ by (i)

   $$= a[X, Y] + b[X, Z]$$ by (a)

   Similarly (ii) $\Rightarrow$ (i)

3. Axiom (c) holds automatically if $\mathrm{dim} \mathfrak{g} = 0, 1, 2$. Now taking these case by case we see,
• dim \( \mathfrak{g} = 0 \Rightarrow \mathfrak{g} = \{0\} \). So we obtain,

\[
3[0, [0, 0]] = 0
\]

but \([0, X] = 0\) for all \( X \in \mathfrak{g} \) and \([0, [0, X]] = 0\). So we obtain,

\[
[X, [aX, bX]] + [aX, [bX, X]] + [bX, [X, aX]] = 3ab[X, [X, X]] = 0
\]

• dim \( \mathfrak{g} = 1 \Rightarrow \mathfrak{g} = \mathbb{F}X \) with \( X \neq 0 \). Now let \( Y \) and \( Z \) be defined as, \( Y = aX \), \( Z = bX \). So we obtain,

\[
[X, [aX, bX]] + [aX, [bX, X]] + [bX, [X, aX]] = 3ab[X, [X, X]] = 0
\]

Example 1.1 (Abelian Lie Algebra). Let \( \mathfrak{g} \) be any vector space with \([X, Y] = 0\) for all \( X, Y \in \mathfrak{g} \). Then axioms (a), (b) and (c) hold trivially.

Example 1.2. Let \( \mathfrak{g} = \text{Mat}_n \mathbb{C} = \{n \times n \text{ matrices over } \mathbb{C}\} \), then \( \dim \mathfrak{g} = n^2 \). We define \([X, Y] = XY - YX\) for all \( X, Y \in \mathfrak{g} \). Then

\[
[Y, X] = YX - XY = -[X, Y],
\]

which is axiom (a). (b) is also easy to check, so I leave it as an exercise. Checking (c), for \( X, Y, Z \in \mathfrak{g} \) we see

\[
X(YZ - ZY) - (YZ - ZY)X
+ Y(ZX - XZ) - (ZX - XZ)Y
+ Z(XY - YX) - (XY - YX)Z = 0
\]

Example 1.3 (Vector Product). Let \( \mathfrak{g} = \mathbb{R}^3 \), where \( \mathbb{F} = \mathbb{R} \). If we let \( u, v \in \mathfrak{g} \) then we can define \([u, v] = u \times v\) (the vector cross product). \([u, v]\) is a Lie bracket!

(a) \( u \times v = -v \times u \)

(b) Known

(c) please check using the identity

\[
v \times (u \times w) = (v \cdot w)u - (v \cdot u)w
\]

Definition: Take any Lie algebra \( \mathfrak{g} \). Then a subset \( \mathfrak{h} \subseteq \mathfrak{g} \) is a Lie subalgebra if:

(a) \( \mathfrak{h} \) is a vector subspace of \( \mathfrak{g} \),

(b) for all \( X, Y \in \mathfrak{h} \) we have \([X, Y] \in \mathfrak{h}\) (i.e. \( \mathfrak{h} \) is closed w.r.t. the Lie bracket).
Remark. Then \( h \) is a Lie algebra (in its own right) when using \([\cdot,\cdot]_g\) as the Lie bracket. This is because the Lie Bracket in \( g \) is a mapping, \([\cdot,\cdot] : h \times h \to h\). As the axioms (a), (b), (c) of a Lie algebra hold for all \( X, Y, Z \in g \) then they will also hold for all \( X, Y, Z \in h \).

Example 1.4 (as in Example 1.2). Let \( g = \text{Mat}_n(\mathbb{C}) \) and define \([X,Y] = XY - YX\). This is the general Lie algebra \( \mathfrak{gl}_n(\mathbb{C}) \).

Example 1.5. The special linear Lie algebra \( \mathfrak{sl}_n(\mathbb{C}) \subset \mathfrak{gl}_n(\mathbb{C}) \) where \( \mathfrak{sl}_n(\mathbb{C}) \) is the set,
\[
\{ X \in \text{Mat}_n(\mathbb{C}) | \text{tr} X = 0 \}.
\]
i.e. if we describe \( X \) as \( X = [x_{ij}]_{i,j=1}^n \) then the condition for \( \text{tr}(X) = 0 \) can be expressed as \( x_{11} + \cdots + x_{nn} = 0 \).

Claim. The Lie algebra \( \mathfrak{sl}_n(\mathbb{C}) \) is a Lie subalgebra in \( \mathfrak{gl}_n(\mathbb{C}) \).

Proof.
(a) Let \( X, Y \in \mathfrak{sl}_n(\mathbb{C}) \) then the \( \text{tr}(X) = 0 = \text{tr}(Y) \). By the linearity of the trace we have
\[
\text{tr}(X + Y) = \text{tr}(X) + \text{tr}(Y) = 0 + 0 = 0.
\]
Thus \( X + Y \in \mathfrak{sl}_n(\mathbb{C}) \). Also if \( X \in \mathfrak{sl}_n(\mathbb{C}) \) we have
\[
\text{tr}(zX) = z \text{tr}(X) = z \cdot 0 = 0
\]
for all \( z \in \mathbb{C} \), so \( zX \in \mathfrak{sl}_n(\mathbb{C}) \). Hence \( \mathfrak{sl}_n(\mathbb{C}) \) is a vector space of \( \mathfrak{gl}_n(\mathbb{C}) \).
(b) Now considering the Lie Bracket. Let \( X, Y \in \mathfrak{sl}_n(\mathbb{C}) \) then
\[
\text{tr}[X,Y] = \text{tr}(XY) - \text{tr}(YX) = 0
\]
which gives us that \([X,Y] \in \mathfrak{sl}_n(\mathbb{C})\). \( \square \)

Example 1.6. \( \mathfrak{gl}_n(\mathbb{C}) \supset \mathfrak{sl}_n(\mathbb{C}) \supset \mathfrak{so}_n(\mathbb{C}) = \{ X \in \text{Mat}_n(\mathbb{C}) | X^t = -X \} \). This is the orthogonal Lie algebra.

(1a) \( \text{tr}(X^t) = \text{tr}(X) \)
(1b) \( \text{tr}(-X) = - \text{tr}(X) \)
Subtracting (1b) from (1a) implies \( \text{tr}(X) = 0 \) and hence \( X \in \mathfrak{sl}_n(\mathbb{C}) \).

Claim. \( \mathfrak{so}_n(\mathbb{C}) \) is a Lie subalgebra of \( \mathfrak{sl}_n(\mathbb{C}) \).

Proof.
(a) We have that \( X, Y \in \mathfrak{so}_n(\mathbb{C}) \) are antisymmetric
\[
(X + Y)^t = X^t + Y^t = -X - Y = -(X + Y),
\]
and hence \( X + Y \in \mathfrak{so}_n(\mathbb{C}) \). For any \( z \in \mathbb{C} \) we have,
(zX)' = z\bar{X} = -zX

which implies that \( zX \in \mathfrak{so}_n(\mathbb{C}) \). Hence \( \mathfrak{so}_n(\mathbb{C}) \) is a vector subspace of \( \mathfrak{sl}_n(\mathbb{C}) \).

(b) Now we examine the Lie Bracket. For \( X, Y \in \mathfrak{so}_n(\mathbb{C}) \)

\[
[X, Y]' = (XY - YX)'
= Y'X' - X'Y'
= YX - XY
= -[X, Y]
\]

\( \therefore [X, Y] \in \mathfrak{so}_n(\mathbb{C}) \) and hence \( \mathfrak{so}_n(\mathbb{C}) \) is a Lie Subalgebra of \( \mathfrak{sl}_n(\mathbb{C}) \).

\[\square\]

Example 1.7 (The Symplectic Lie algebra). Let \( Y \in \text{Mat}_{2n}(\mathbb{C}) \). Then \( Y \) is a matrix of the form

\[
\begin{pmatrix}
P & Q \\
R & S
\end{pmatrix}
\]

where \( P, Q, R \) and \( S \) are \( n \times n \) matrices. Then we define the Symplectic Group to be the set

\[
\mathfrak{sp}_{2n}(\mathbb{C}) = \{ X \in \text{Mat}_{2n}(\mathbb{C}) \mid S = -P^t, R^t = R, Q^t = Q \}.
\]

Check this is a Lie subalgebra in \( \mathfrak{gl}_{2n}(\mathbb{C}) \).

Definition: Let \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) be any two Lie algebras over \( \mathbb{F} (= \mathbb{C}) \). Then a Lie algebra homomorphism \( \phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2 \) is a map (or transformation) \( \phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2 \), such that

(a) \( \phi \) is a linear map, \( \phi(aX + bY) = a\phi(X) + b\phi(Y) \) with \( a, b \in \mathbb{F} \) and \( X, Y \in \mathfrak{g}_1 \).

(b) \( \phi([X, Y]_{\mathfrak{g}_1}) = [\phi(X), \phi(Y)]_{\mathfrak{g}_2} \).

Example 1.8 (Trivial). For all \( \mathfrak{g}_1, \mathfrak{g}_2 \) Lie algebras \( \phi(X) \equiv 0 \) defines a Lie algebra homomorphism.

Definition: \( \phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2 \) is a Lie algebra isomorphism if \( \phi \) is a homomorphism and \( \phi \) is bijective (one-to-one and onto).

Remark. If \( \phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2 \) is an isomorphism then we can identify \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) by identifying \( X \) and \( \phi(X) \) for all \( X \in \mathfrak{g}_1 \). If such an isomorphism exists between \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) then we say the two Lie algebras are isomorphic (essentially the same). We will study Lie algebras up to their isomorphisms.

Example 1.9. Let \( V \) be a vector-space over \( \mathbb{C} \) such that \( \text{dim} V = n \). Then we have a set \( \mathfrak{gl}(V) = \{ T : V \rightarrow V \} \), which contains all the possible mappings from \( V \) to itself. We define a Lie bracket for \( T, S \in \mathfrak{gl}(V) \) as,

\[
[T, S] = TS - ST
\]
Proof. Same as in Example 1.2. □

Claim. $\mathfrak{gl}(V)$ is isomorphic to $\mathfrak{gl}_n(\mathbb{C})$. i.e. Any linear map can be expressed as a matrix.

Proof. Choose any basis $\{v_1, \ldots, v_n\}$ in $V$. For all $T \in \mathfrak{gl}(V)$ consider its matrix $X = X_T = [x_{ij}]_{i,j=1}^n$. Now we can describe the mapping onto the basis as,

$$(2) \quad Tv_j = \sum_{i=1}^n x_{ij}v_i, \quad \forall j = 1, \ldots, n.$$ 

Define $\varphi : \mathfrak{gl}(V) \rightarrow \mathfrak{gl}_n(\mathbb{C})$ to be the map $\varphi(T) = X_T$. We need to check that $\varphi$ is a Lie algebra isomorphism.

• First we check the homomorphism property. We can see $\varphi$ is linear, i.e. $\varphi(aT + bS) = a\varphi(T) + b\varphi(S)$. So,

$$\varphi([T, S]_{\mathfrak{gl}(V)}) = \varphi(TS - ST)$$
$$= \varphi(TS) - \varphi(ST)$$
$$= \varphi(T)\varphi(S) - \varphi(S)\varphi(T)$$
$$= [\varphi(T), \varphi(S)]_{\mathfrak{gl}_n(\mathbb{C})}$$

See Linear Algebra for verification of these statements.

• $\varphi$ is bijective by (2). □

Remark. $\varphi$ depends on the choice of a basis.

Remark. Examining special cases of $\mathfrak{gl}_n(\mathbb{C})$ we see

• $n = 1$: $\mathfrak{gl}_1(\mathbb{C}) = \{x \in \mathbb{C}\}$ and $[x, y] = 0$. Hence abelian.

• $n = 2$: $\mathfrak{gl}_2(\mathbb{C}) = \{x \in \mathbb{C}\}$ is non-Abelian, i.e. there exists $X, Y$ such that $XY \neq YX$.

Now examining similar cases $\mathfrak{sl}_n(\mathbb{C})$ we see

• $n = 1$: $\mathfrak{sl}_1(\mathbb{C}) = \{x \in \mathbb{C}\}$ and dim $\mathfrak{sl}_1(\mathbb{C}) = 0$.

• $n = 2$: $\mathfrak{sl}_2(\mathbb{C}) = \{\begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{C}\}$ and dim $= 3$.

Now examining similar cases for $\mathfrak{so}_n(\mathbb{C}) \subset \mathfrak{sl}_n(\mathbb{C})$ we see

• $n = 1$: $\mathfrak{so}_1(\mathbb{C}) = \{0\}$, dim $= 0$

• $n = 2$: $\mathfrak{so}_2(\mathbb{C}) = \{\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} : x \in \mathbb{C}\}$. The Lie bracket gives us,

$$[X, Y] = XY - YX = \begin{pmatrix} -xy & 0 \\ 0 & -xy \end{pmatrix} - \begin{pmatrix} -yx & 0 \\ 0 & -yx \end{pmatrix} = 0.$$ 

• $n = 3$, $\mathfrak{so}_3(\mathbb{C}) = \{\begin{pmatrix} 0 & y & z \\ -y & 0 & -z \\ -z & y & 0 \end{pmatrix} : x, y, z \in \mathbb{C}\}$. We have that dim $\mathfrak{so}_3(\mathbb{C}) = 3$ and non-Abelian!

**Proposition:** $\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{so}_3(\mathbb{C})$

Remark. $\mathfrak{sl}_2(\mathbb{R}) \ncong \mathfrak{so}_3(\mathbb{R})$, so the field $\mathbb{F}$ of our Lie algebra does matter.
Proof. Define \( \varphi : \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathfrak{so}_3(\mathbb{C}) \) such that

\[
\varphi \left( \begin{array}{cc} a & b \\ c & -a \end{array} \right) = \left( \begin{array}{ccc} 0 & b - c & -i(b + c) \\ c - b & 0 & 2ai \\ i(b + c) & -2ai & 0 \end{array} \right)
\]

We need to show that \( \varphi \) is a Lie algebra isomorphism. First we check to see if \( \varphi \) is a bijection (i.e. one-to-one and onto).

**One-to-one:** We know \( \varphi(X) \) and we know \( X \). We can clearly see that \( \varphi(X_1) = \varphi(X_2) \Leftrightarrow X_1 = X_2 \) and so \( \varphi \) is one-to-one.

**Onto:** Let \( Z \) be any element of \( \mathfrak{so}_3(\mathbb{C}) \), i.e. \( \left( \begin{array}{ccc} 0 & z_1 & z_2 \\ -z_1 & 0 & z_3 \\ -z_2 & -z_3 & 0 \end{array} \right) \) then we can find \( X \in \mathfrak{sl}_2(\mathbb{C}) \) such that \( \varphi(X) = Z \). Specifically,

\[
\begin{align*}
z_1 &= b - c \\
z_2 &= -i(b + c) \\
z_3 &= 2ia \\
a &= \frac{z_3}{2i} \\
b &= \frac{z_2 - iz_1}{-2i} \\
c &= \frac{iz_1 + z_2}{-2i}
\end{align*}
\]

\( \Rightarrow X = \left( \begin{array}{ccc} \frac{z_3}{2i} & \frac{z_2 - iz_1}{-2i} & \frac{-2i}{z_3} \\ iz_1 + z_2 & \frac{-2i}{z_2} & \frac{z_3}{2i} \\ -z_2 & -z_3 & 0 \end{array} \right) \)

So \( \varphi \) is a bijection.

Finally we check the properties of a homomorphism. The entries of \( \varphi(X) \) are linear functions in \( a, b, c \) therefore this means \( \varphi \) is a linear map, this is the first property of a homomorphism satisfied. Left to check that

\[
(3) \quad \varphi[X, X'] = [\varphi(X), \varphi(X')].
\]

Now define \( X' \in \mathfrak{sl}_2(\mathbb{C}) \) to be \( X' = \left( \begin{array}{cc} a' & b' \\ c' & -a' \end{array} \right) \). Then the commutator gives us,

\[
[X, X'] = XX' - X'X
= \left( \begin{array}{cc} a & b \\ c & -a \end{array} \right) \left( \begin{array}{cc} a' & b' \\ c' & -a' \end{array} \right) - \left( \begin{array}{cc} a' & b' \\ c' & -a' \end{array} \right) \left( \begin{array}{cc} a & b \\ c & -a \end{array} \right)
= \left( \begin{array}{cc} aa' + bc' & ab' - ba' \\ ca' - ac' & cb' + aa' \end{array} \right) - \left( \begin{array}{cc} a'a + b'c & a'b - b'a \\ c'a - a'c & cb' + a'a \end{array} \right)
\]

Now returning to (3) we get the left hand side of the equation to be

\[
\varphi([X, X']) = \left( \begin{array}{ccc} 0 & ab' - ba' - a'b + b'a & -i(ab' - b'a - a'b + b'a) \\ -ca' + ac' + c'a - ac' & +ca' - a'c - c'a + a'c & 2i(aa' + bc' - a'a + b'c) \\ * & 0 & * \end{array} \right)
\]

and the right hand side of the equation to be
\[
\begin{align*}
\left[\varphi(X), \varphi(X')\right] &= \varphi(X)\varphi(X') - \varphi(X')\varphi(X) \\
&= \begin{pmatrix}
0 & b - c & -i(b + c) \\
(c - b) & 0 & 2ai \\
i(b + c) & -2ai & 0
\end{pmatrix}
\begin{pmatrix}
0 & b' - c' & -i(b' + c') \\
c' - b' & 0 & 2a'i \\
i(b' + c') & -2a'i & 0
\end{pmatrix} \\
&= \varphi(X')\varphi(X) - \varphi(X)\varphi(X') \\
&= \text{LHS}
\end{align*}
\]

This method is extremely ugly! We need a more sophisticated method to construct isomorphisms between two Lie algebras.

**Definition:** We define the *direct sum* of two Lie algebras \(g_1, g_2\) over \(F\) to be

\[
g_1 \oplus g_2 = \{(X_1, X_2) : X_1 \in g_1, X_2 \in g_2\},
\]

with component wise addition and scalar multiplication. The Lie bracket for the direct sum is

\[
[(X_1, X_2), (Y_1, Y_2)]_{g_1 \oplus g_2} \equiv ([X_1, Y_1]_{g_1}, [X_2, Y_2]_{g_2})
\]

We claim, without proof, that the axioms of a Lie algebra are satisfied by the direct sum.

**Remark** (on Homomorphisms). We have seen that \(\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{so}_3(\mathbb{C})\). Also, one can prove (although difficult) that \(\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{so}_4(\mathbb{C})\) and \(\mathfrak{sp}_4(\mathbb{C}) \cong \mathfrak{so}_5(\mathbb{C})\) and \(\mathfrak{sl}_4(\mathbb{C}) \cong \mathfrak{so}_6(\mathbb{C})\). These are all isomorphisms for \(\mathfrak{so}_n(\mathbb{C})! \ (n > 6)\)

**Definition:** Take any Lie algebra \(\mathfrak{g}\) (over \(F\)). Then the *derived Lie algebra* (of \(\mathfrak{g}\)) is the Lie subalgebra \(\mathfrak{g}' \subset \mathfrak{g}\) such that

\[
\mathfrak{g}' = \text{span} \{ [X, Y]_{\mathfrak{g}} : X, Y \in \mathfrak{g} \}
\]

which has typical element \(a_1[X_1, Y_1] + a_2[X_2, Y_2] + \ldots\)

- \(P, Q \in \mathfrak{g}' \Rightarrow P, Q \in \mathfrak{g}\)

1.1. Derivation of Lie Algebras

**Definition:** a *derivation* of a Lie algebra \(\mathfrak{g}\) is a linear transformation (operator) \(\delta : \mathfrak{g} \rightarrow \mathfrak{g}\) satisfying the Liebniz Rule,

\[
\delta([X, Y]) = [\delta(X), Y] + [X, \delta(Y)] \quad \text{for all} \ X, Y \in \mathfrak{g}
\]
Caution: a derivation $\delta$ of $\mathfrak{g}$ should NOT be confused with the derived Lie algebra $\mathfrak{g}'$.

Notation: the set of all derivations of $\mathfrak{g}$ is denoted by $\mathcal{D}(\mathfrak{g}) = \{\delta \mid \delta \text{ is a derivation}\}$.

Definition: For all $Z \in \mathfrak{g}$ we define the adjoint operator $\text{ad} Z : \mathfrak{g} \to \mathfrak{g}$ to be the mapping $\text{ad} Z(X) = [Z, X]$.

Claim. For all $Z \in \mathfrak{g}$ we have $\text{ad} Z \in \mathcal{D}(\mathfrak{g})$.

Proof. To show that this is true we have to show that the adjoint operator is linear and satisfies the Liebniz rule. So, let $a, b \in \mathbb{F}$ and $X, Y \in \mathfrak{g}$.

- We first show it is linear
  
  $$\text{ad} Z(aX + bY) = [Z, aX + bY]$$
  
  $$= a[Z, X] + b[Z, Y]$$
  
  $$= a \times \text{ad} Z(X) + b \times \text{ad} Z(Y)$$

- Now we have to prove that the Liebniz rule holds.

  $$(\text{ad} Z)[X, Y] = [Z, [X, Y]]$$
  
  $$= -[X, [Y, Z]] - [Y, [Z, X]]$$
  
  by Jacobi identity
  
  $$= [X, [Z, Y]] + ([Z, X], Y)$$
  
  $$= [X, \text{ad} Z(Y)] + [\text{ad} Z(X), Y]$$
  
  $$= [\text{ad} Z(X), Y] + [X, \text{ad} Z(Y)]$$.

Hence the Liebniz rule holds for the adjoint operator and we have proved it is in $\mathcal{D}(\mathfrak{g})$. \qed

Definition: Let $\mathfrak{g}$ be any Lie algebra. Then we define the centre to be the set

$$\mathcal{Z}(\mathfrak{g}) = \{Z \in \mathfrak{g} \mid [Z, X] = 0 \text{ for all } X\}$$

or equivalently we can say $Z \in \mathcal{Z}(\mathfrak{g}) \iff \text{ad} Z \equiv 0$.

Example 1.10. Let $\mathfrak{g}$ be an abelian Lie algebra, i.e. $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. Then $\mathcal{Z}(\mathfrak{g}) = \mathfrak{g}$.

Example 1.11. Let us look at the general linear Lie algebra $\mathfrak{gl}_n(\mathbb{C})$. Then the centre is,

$$\mathcal{Z}(\mathfrak{gl}_n(\mathbb{C})) = \{\text{scalar multiples of the } n \times n \text{ identity matrix}\}$$

$$= \left\{ \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} ; \lambda \in \mathbb{C} \right\}$$
Example 1.12. It is now logical to look at the special linear Lie algebra \( \mathfrak{sl}_n(\mathbb{C}) \). We have that \( Z(\mathfrak{sl}_n(\mathbb{C})) = \{0\} \). It is easy to see that 0 is in the centre, \( [0, X] = 0 \Rightarrow 0 \in Z(\mathfrak{g}) \), but we shall prove later that it is the only element in the centre.

**Theorem 1.** \( \mathcal{D}(\mathfrak{g}) = \{ \delta \mid \delta \) is a derivation of \( \mathfrak{g} \} \) is a Lie algebra with addition / scalar multiplication as those of linear transformations and the Lie bracket being the commutator \( [\delta_1, \delta_2] = \delta_1 \delta_2 - \delta_2 \delta_1 \).

**Proof.** We need to check all the axioms of a Lie algebra for \( \mathcal{D}(\mathfrak{g}) \).

- Is \( \mathcal{D}(\mathfrak{g}) \) a vector space? For \( \delta_1, \delta_2 \in \mathcal{D}(\mathfrak{g}) \) then \( \delta_1 + \delta_2 \) is another linear map \( (\delta_1 + \delta_2) : \mathfrak{g} \rightarrow \mathfrak{g} \), such that

\[
(\delta_1 + \delta_2)(X) = \delta_1(X) + \delta_2(X) \quad \text{for all } X \in \mathfrak{g}.
\]

- Does the Leibniz rule still hold?

\[
(\delta_1 + \delta_2) = \delta_1[X,Y] + \delta_2[X,Y]
\]

\[
= [\delta_1(X), Y]_\mathfrak{g} + [X, \delta_1(Y)]_\mathfrak{g} + [\delta_2(X), Y]_\mathfrak{g} + [X, \delta_2(Y)]_\mathfrak{g}
\]

\[
= [(\delta_1 + \delta_2)(X), Y] + [X, (\delta_1 + \delta_2)(Y)].
\]

Yes it does. Similarly we can check that for all \( \delta \in \mathcal{D}(\mathfrak{g}) \) and for all \( a \in \mathbb{F} (= \mathbb{C}) \) then \( a\delta \in \mathcal{D}(\mathfrak{g}) \) where,

\[
(a\delta)(X) = a(\delta(X))
\]

- Is \( [\delta_1, \delta_2] = \delta_1 \delta_2 - \delta_2 \delta_1 \) a Lie bracket? Also is \( [\delta_1, \delta_2] \in \mathcal{D}(\mathfrak{g}) \)? We show this by checking that the Liebniz rule holds for \( \delta_1 \delta_2 - \delta_2 \delta_1 \).

\[
(\delta_1 \delta_2 - \delta_2 \delta_1)[X,Y]_\mathfrak{g} = \delta_1(\delta_2[X,Y]) - \delta_2(\delta_1[X,Y])
\]

\[
= \delta_1 \left( [\delta_2(X), Y] + [X, \delta_2(Y)] \right) - \delta_2 \left( [\delta_1(X), Y] + [X, \delta_1(Y)] \right)
\]

\[
= [(\delta_1 \delta_2)(X), Y] + [\delta_2(X), \delta_1(Y)] + [\delta_1(X), \delta_2(Y)] + [X, (\delta_1 \delta_2)(Y)]
\]

\[
- [(\delta_2 \delta_1)(X), Y] - [\delta_1(X), \delta_2(Y)] - [\delta_2(X), \delta_1(Y)] - [X, (\delta_2 \delta_1)(Y)]
\]

\[
= [(\delta_1 \delta_2 - \delta_2 \delta_1)(X), Y] + [X, (\delta_1 \delta_2 - \delta_2 \delta_1)(Y)].
\]

So this means the commutator \( \delta_1 \delta_2 - \delta_2 \delta_1 \) of two derivatives \( \delta_1, \delta_2 \) is a derivation again and hence \( [\delta_1, \delta_2] \in \mathcal{D}(\mathfrak{g}) \). Checking the axioms (a), (b), (c) for the commutator \( \delta_1 \delta_2 - \delta_2 \delta_1 \) is similar to checking the axioms for the matrix commutator, \( AB - BA \) where \( A, B \in \text{Mat}(\mathbb{C}) \). \( \square \)

We define a function “ad” such that

\[
ad : \mathfrak{g} \rightarrow \mathcal{D}(\mathfrak{g})
\]

\[
Z \rightarrow \text{ad} Z,
\]
where $Z \in \mathfrak{g}$ and $\text{ad} Z \in \mathcal{D}(\mathfrak{g})$.

Proposition. “$\text{ad}$” is a Lie algebra homomorphism.

Proof. We need to show that the “$\text{ad}$” operator is linear, i.e. satisfies the condition $\text{ad}(pX + qY) = p\text{ad} X + q\text{ad} Y$. We show this by applying the identity to any $Z \in \mathfrak{g}$.

$$\text{ad}(pX + qY)(Z) = [pX + qY, Z]$$
$$= [pX, Z] + [qY, Z]$$
$$= p[X, Z] + q[Y, Z]$$
$$= p\text{ad}(X)(Z) + q\text{ad}(Y)(Z)$$

for all $X, Y \in \mathfrak{g}$. However we also need to show that $\text{ad}([X, Y]_\mathfrak{g}) = [\text{ad} X, \text{ad} Y]_\mathcal{D}(\mathfrak{g})$. We show this by again applying both sides to any $Z \in \mathfrak{g}$. This gives,

$$\text{ad} (\{X, Y\}_\mathfrak{g})(Z) = \{[X, Y]_\mathfrak{g}, Z\}_\mathfrak{g}$$
$$= \{X, [Y, Z]_\mathfrak{g}\}_\mathfrak{g} + \{[Z, X]_\mathfrak{g}, Y\}_\mathfrak{g}$$
$$= \{X, \text{ad}(Y)(Z)\}_\mathfrak{g} - \{Y, \text{ad}(X)(Z)\}_\mathfrak{g}$$
$$= ([\text{ad} X](Y) - \text{ad} Y)(\text{ad} X)(Z)$$
$$= \text{ad}(X), \text{ad}(Y)]_\mathcal{D}(\mathfrak{g})(Z).$$

The second line is obtained using the Jacobi identity and this holds for all $X, Y \in \mathfrak{g}$. □

Notation: We use the following notation to describe the Lie bracket of two Lie algebras. So, for $\mathfrak{g}_1, \mathfrak{g}_2$ we have

$$[\mathfrak{g}_1, \mathfrak{g}_2]_\mathfrak{g} \equiv \text{span} \{[X_1, X_2] : X_1 \in \mathfrak{g}_1, X_2 \in \mathfrak{g}_2\}.$$

Note that in this notation we have $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$.

Definition:

1. a derivation $\delta$ is called inner if $\delta = \text{ad} Z$ for some $Z \in \mathfrak{g}$.
2. if $\delta$ is not inner, then it is called an outer derivation.

$$\mathcal{D}(\mathfrak{g}) = \{\text{inner}\} \cup \{\text{outer}\}$$

Example 1.13 (Very Important Example of a Lie Algebra). Consider a Lie algebra $\mathfrak{g}$ of dimension 2 with basis $\{E, F\}$. Define a Lie bracket on $\mathfrak{g}$ by $[E, E] = 0$, $[F, F] = 0$ and $[E, F] = E \Rightarrow [F, E] = -E$. It can be extended by bi-linearity to

$$[aE + bF, cE + dF] = adE - bcE.$$
Observation: \( \mathfrak{g} \) is not Abelian.

**Proposition.** Every derivation of this \( \mathfrak{g} \) is inner.

**Proof.** Take any derivation \( \delta \in \mathcal{D}(\mathfrak{g}) \). We need to prove that \( \delta = \text{ad} Z \) for some \( Z \in \mathfrak{g} \).

1. We can see that the derived Lie algebra is \( \mathfrak{g}' = CE \). If \( \delta(\mathfrak{g}') \subset \mathfrak{g}' \) then \( \delta(E) = dE \) for some \( d \in \mathbb{C} \).
2. If we apply \( \text{ad}(dF) \) to \( E \) then we obtain, \[
\text{ad}(dF)(E) = \left[ dF, E \right] = -dE.
\]
   Now define \( \epsilon \in \mathcal{D}(\mathfrak{g}) \) to be \( \epsilon = \delta + \text{ad}(dF) \).

3. Using the definition of the Lie brackets we can obtain that, \[
0 = \epsilon(E) = \epsilon\left( [E, F] \right) = [\epsilon(E), F] + [E, \epsilon(F)].
\]
   \( \therefore [E, \epsilon(F)] = 0 \). Now looking at \( \epsilon \) applied to \( F \) we have that \( \epsilon(F) \in \mathfrak{g} \Rightarrow \epsilon(F) = xE + yF \). Using the previous result we get \[
0 = [E, \epsilon(F)] = [E, xE + yF] = yE,
\]
   this implies that \( y = 0 \Rightarrow \epsilon(F) = xE \).

4. Now consider \( \text{ad}(xE) \) and apply it to \( E \). This gives us \( \text{ad}(xE)(E) = [xE, E] = 0 \).
   Looking at the operator applied to \( F \) gives \( \text{ad}(xE)(F) = [xE, F] = xE \).
   However we have \( \epsilon(E) = 0 \) by step 2 and \( \epsilon(F) = xE \) by step 3. We have two linear transformations, \( \text{ad}(xE) \) and \( \epsilon \), that are acting in the same way on the basis vectors \( E, F \). So this means we must have \( \text{ad}(xE) = \epsilon \).

5. Then \( \text{ad}(xE) = \delta + \text{ad}(dF) \Rightarrow \delta = \text{ad}(x\overline{E} - d\overline{F}) \).

**Definition:** For any Lie algebra \( \mathfrak{g} \), a subset \( \mathfrak{h} \subseteq \mathfrak{g} \) is an **ideal** if:

- \( \mathfrak{g} \supseteq \mathfrak{h} \) is a vector subspace,
- \( [X, Y] \in \mathfrak{h} \) for all \( X \in \mathfrak{g}, Y \in \mathfrak{h} \).

**Remark.** If \( \mathfrak{h} \) is an ideal then this implies \( \mathfrak{h} \) is a Lie subalgebra of \( \mathfrak{g} \) (we have from the definition that \( [X, Y] \in \mathfrak{h} \) for all \( X, Y \in \mathfrak{h} \)), but this is not true the other way round. Consider Example 1.13. In that example take \( \mathfrak{h} = CE \). This is an ideal,

\[
\mathfrak{g} \ni [X, E] = [aE + bF, E] = -bE \in \mathfrak{h}.
\]
Now consider \( \mathfrak{f} = CF \). This is a subalgebra \( [aF, bF] = 0 \). Now \( [aE + bF, F] = aE \) for any \( a \neq 0 \) therefore \( [aE + bF, F] \notin \mathfrak{f} \). Hence \( \mathfrak{f} \) is not an ideal.

**Definition:** A Lie algebra \( \mathfrak{g} \) is called **complete** if

1. \( \mathcal{Z}(\mathfrak{g}) = \{0\} \)
and every derivation of \( g \) is inner.

**Example 1.14** (Example 1.13 revisited). We have already proved part (2) of the definition of complete for this example. So, now let us check part (1). Suppose \( Z \in Z(g) \) and define \( Z \) to be \( Z = aE + bF \). Then

\[
0 = [Z, E] = [aE + bF, E] = -bE \Rightarrow b = 0
\]

\[
0 = [Z, F] = [aE + bF, F] = aE \Rightarrow a = 0
\]

therefore \( Z = 0 \Rightarrow Z(g) = \{0\} \).

**Theorem 2.** Let \( g \) be any Lie algebra. Suppose that \( h \subseteq g \) is an ideal, which is complete as a Lie algebra (in its own right). Then there exists another ideal \( f \subseteq g \) such that \( g = h \oplus f \).

**Proof.** Let \( f = \{ Z \in g \mid [X, Z] = 0 \text{ for all } X \in h \} \), i.e. the “centraliser of \( h \) in \( g \)”. We first need to show that \( f \) is a Lie subalgebra of \( g \). This has two parts.

- Is \( f \) a vector subspace? Well we have that for \( Z_1, Z_2 \in f \) that \( Z_1 + Z_2 \in f \) by linearity of \([ , , ]\). Also for all \( Z \in f \), \( \alpha \in \mathbb{C} \) then \( \alpha Z \in f \). So \( f \) is a vector subspace of \( g \).
- Now for all \( Y \in g \) and \( Z \in f \) we need to show that \([Y, Z] \in f \), i.e. \([X, [Y, Z]] = 0 \text{ for all } X \in h \). By the Jacobi identity we have,

\[
[X, [Y, Z]] = -[Y, [Z, X]] - [Z, [X, Y]].
\]

We have \([X, Y] \in h \) because it’s a \( h \) ideal, which implies \([Z, [X, Y]] = 0 \) by definition of \( f \). Also by the definition of \( f \) we have \([Z, X] = 0 \), which implies \([Y, [Z, X]] = 0 \). Hence \([X, [Y, Z]] = 0 \) as required.

So \( f \) is a Lie subalgebra of \( g \). We now need to show that \( g = h \oplus f \) is a Lie algebra.

1. Start by proving that \( h \cap f = \{0\} \). Take any \( Z \in h \cap f \). This means \( Z \in f \Rightarrow [Z, X] = 0 \text{ for all } X \in h \) but \( Z \) is also in \( h \). Therefore \([Z, X] = 0 \text{ for all } X \in h \) implies that \( Z \in Z(h) \). We know \( h \) is complete so we must have \( Z = 0 \).

2. We now show that \( g = h + f \) (which is equal to \( h \oplus f \) because \( h \cap f = \{0\} \)). Take any \( A \in g \) and consider the operator;

\[
ad A : g \to g.
\]

For all \( X \in h \) we have that \( \text{ad} A(X) = [A, X] \), which is an element of \( h \) because \( h \) is an ideal. Therefore we have that \( \text{ad} A \) will map any element of \( h \) to \( h \), i.e. \( \text{ad} A : h \to h \). The ‘ad’ operator satisfies the Liebniz rule and hence \( \text{ad} A \in D(h) \). Apriori we may not know that \( \text{ad} A \) is an inner derivation because \( A \) may be outside \( h \). However \( h \) is complete as a Lie algebra, so any derivation is inner. So \( \text{ad} A|_h \) is an inner derivation. Now comparing \( \text{ad} A \) with another operator from the \( D(h) \) we see,
\[(\text{ad } A)(X) = (\text{ad } B)(X) \quad \text{for some } B \in \mathfrak{h}\]
\[[A, X] = [B, X] \quad \text{for all } X \in \mathfrak{h}\]
\[[A - B, X] = 0 \quad \text{for all } X \in \mathfrak{h}\]

hence \(A - B \in \mathfrak{f}\) by definition of \(\mathfrak{f}\). Now defining \(C = A - B\), we have that \(A = B + C\) for any \(A \in \mathfrak{g}, B \in \mathfrak{h}, C \in \mathfrak{f}\).

(3) Now only left to prove that \(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{f}\) is a Lie algebra. We do this by constructing an isomorphism,

\[\varphi : \mathfrak{g} \rightarrow \mathfrak{h} \oplus \mathfrak{f}\]
\[A \mapsto (B, C)\]

a trivial isomorphism. We know that \(\varphi\) is bijective and linear by definition of vector spaces. Need to show that \(\varphi\) satisfies the homomorphism property,

\[\varphi([A, A']_{\mathfrak{g}}) = \varphi(A), \varphi(A')_{\mathfrak{h} \oplus \mathfrak{f}}\]

where \(A \mapsto (B, C), A' \mapsto (B', C')\) and \(A = B + C, A' = B' + C'\). We start with \(B, B' \in \mathfrak{h}\) and \(C, C' \in \mathfrak{f}\) then

\[\varphi([B + C, B' + C']) = \varphi([B, B'] + [C, C'])\]
\[= \varphi([B, B']) + \varphi([C, C'])\]

because \([B, B'] \in \mathfrak{h}\) and \([C, C'] \in \mathfrak{f}\) we have that \(\varphi\) maps them to,

\[= ([B, B'], 0) + (0, [C, C'])\]
\[= ([B, B'], [C, C'])\]
\[= [(B, C), (B', C')]\]
\[= [\varphi(B + C), \varphi(B' + C')]\]

\[\square\]

2. **Classification of Lie algebras of dimensions 1, 2, 3**

For the next couple of lectures we will be trying to classify Lie algebras of small dimension. This will help us a lot in the theory of Lie algebras. We will be focusing on Lie algebras of dimension 1, 2 or 3 and will take them case by case.

1) Take a Lie algebra \(\mathfrak{g}\) such that \(\text{dim } \mathfrak{g} = 1\). Let \(X, Y \in \mathfrak{g} = \mathbb{C}E\) where \(E \in \mathfrak{g}\) and \(E \neq 0\) then we must have \(X = xE, Y = yE\). So the Lie bracket of these two elements is,

\[[X, Y] = [xE, yE] = xy[E, E] = 0.\]

This is an abelian Lie algebra and hence for one dimension we must have \([ , ] \equiv 0\).
II) Take a Lie algebra $g$ such that $\dim g = 2$. We consider the possible cases of $g' \subset g$ but in fact there are only two.

a) $g' = \{0\}$, which from above gives us $[X, Y] = 0$ for all $X, Y \in g$ and hence $g$ is an abelian Lie algebra.

b) $g' \neq \{0\}$. Then $g'$ must be the span of two basis vectors (as $\dim g = 2$). So, choose any basis in $g$, say $\{E, F\}$. Then,

$$g' = \text{span} \{[X, Y] | X, Y \in g \}$$

$$= \text{span} \{[E, E], [E, F], [F, E], [F, F]\}$$

$$= \text{span} \{[E, F]\}$$

by $[E, E] = [F, F] = 0$

$$= C[E, F]$$

but $g' \neq 0$ by assumption and hence $[E, F] \neq 0$. Also we note that $\dim g' = 1$. Up to this point our choice of $E, F$ was arbitrary, however let us now choose a specific $E$ and $F$. We choose $E$ and $F$ such that $g' = CE$. This means that $[E, F] = cE$ for some $c \in C \setminus \{0\}$. Thus we have,

$$\begin{bmatrix} E & F \\ E & \frac{F}{c} \end{bmatrix} = E$$

but we can just set $F$ to be $\frac{E}{c}$. Doing this gives us $[E, F] = E$. Now $g$ has a basis, $\{E, F\}$, which corresponds to the Lie bracket relationships,

$$[E, F] = 0 \quad [E, F] = E$$

$$[F, F] = 0 \quad [F, E] = -E.$$

We notice that this is our Very Important Example, i.e. Example 1.13. So, this set of Lie algebras are the only non-abelian Lie algebras of 1 or 2 dimensions.

III) Take a Lie algebra $g$ such that $\dim g = 3$. We will again consider $g'$ and its various cases.

(a) $\dim g' = 0$ then $g' = \{0\}$ and hence $g$ is an abelian Lie algebra.

(b) $\dim g' = 1$ and let us assume that $g' \subseteq Z(g)$ (the centre of $g$). Now choose an element $E \in g$ such that $g' = CE$ with $E \neq 0$. We now complete $E$ to a basis, $\{E, F, G\}$ in $G$. So we have $E \in g' \subseteq Z(g)$ and hence $[E, X] = 0$ for all $X \in g$. Then, again using that $E \in Z(g)$ we obtain the Lie bracket relationships;

$$[E, F] = [E, G] = 0.$$

We know $g'$ is a span of the basis so we obtain,
\[ g' = \text{span}\{ [E, F], [E, G], [F, G] \} \]
\[ = \text{span}\{ [F, G] \} \]
\[ = \mathbb{C}E. \]

So we must have that \([F, G] = dE\) for some \(d \neq 0 \in \mathbb{C}\). Just as before we obtain a “new” \(E\) by setting \(E\) to \(dE\). We now obtain our final Lie bracket relationship to be

\[ [F, G] = E \]

(N.B. this also gives us \([G, F] = -E\) to satisfy antisymmetry). Hence there is only one Lie algebra \(g\) such that \(\text{dim} \ g = 3\), \(\text{dim} \ g' = 1\) and \(g' \subseteq \mathcal{Z}(g)\).

**Exercise:** it is still left to check that (4) and (5) define a Lie algebra.

(c) \(\text{dim} \ g' = 1\) and let us assume this time that \(g' \not\subseteq \mathcal{Z}(g)\). Now choose an element \(E \in g\) such that \(g' = \mathbb{C}E\) with \(E \neq 0\). We assume \(E \not\in \mathcal{Z}(g)\) as it implies that there exists an \(F \in g\) such that \([E, F] \neq 0\) otherwise \([E, F] = 0\) for all \(F \in g\).

Also we need \(F \neq xE\) otherwise

\[ [E, F] = [E, xE] = x[E, E] = x.0 = 0. \]

So we use \(E, F\) as a two basis vector. By the definition of \(g'\) we know \([E, F] \in g' = \mathbb{C}E\) and hence \([E, F] = bE\) for some, non-zero, \(b \in \mathbb{C}\). Just as before we have that the Lie bracket of \(E\) and \(F\) will be

\[ \left[ E, \frac{F}{b} \right] = E. \]

So setting \(F\) to \(\frac{F}{b}\) we get the Lie bracket relationship \([E, F] = E\).

Now consider the Lie subalgebra \(h \subset g\) spanned by \(E, F\). From above we have that \([E, F] = E\), \([E, E] = [F, F] = 0\) but this is just our Very Important Example, i.e. Example 1.13. Moreover, \(h\) is an ideal of \(g\) for all \(X \in g\) and for all \(Y \in h\). By definition we have that \([X, Y] \in g' = \mathbb{C}E \subset h\). However, by Example 1.14, we know that \(h\) is complete as a Lie algebra. Thus by Theorem 2 we have that there exists an \(f \subset g\) such that \(g = h \oplus f\). Looking at the dimensions of our Lie algebras we see that

\[ \text{dim} \ g = \text{dim} \ h + \text{dim} \ f \]
\[ 3 = 2 + ? \]

Hence \(f\) must have dimension 1 and by above results is abelian. Hence we must have for some \(G \in g\) that \(f = \mathbb{C}G\). Now \(\{E, F, G\}\) is a basis of \(g\), with Lie bracket relationships,
\[ [E, F] = E \quad [E, G] = 0 \quad [F, G] = 0. \]

We note again that there is only one Lie algebra \( g \) with \( \dim g = 3 \), \( \dim g' = 1 \) and \( g' \not\subseteq Z(g) \).

(d) \( \dim g' = 2 \). We start by proving a useful remark about \( g' \).

**Remark.** \( g' \) **MUST** be abelian.

**Proof.** Suppose, for a contradiction, that \( g' \) is not Abelian. Then \( g' \subset g \) is always an ideal of \( g \). So, for all \( X \in g, Y \in g' \) we have that \( [X, Y] \in g' \) and by example 1.14 \( g' \) is complete. Then by Theorem 2 we can express \( g \) as,

\[ g = g' \oplus f = (CE \oplus CF) \oplus CG, \]

letting \( f = CG \). However, \( g' \) is also

\[ g' = \text{span}\{[X, Y] | X, Y \in g\} = \text{span}\{[E, F], [E, G], [F, G]\} = \text{span}\{E\} \]

by the Lie bracket relationships in part (c), thus \( \dim g' = 1 \) but this is a contradiction. Hence \( g' \) must be abelian. \( \square \)

Now choose any basis in \( g' \), say \( \{E, F\} \) then \( [E, F] = 0 \) by above remark. We complete \( \{E, F\} \) to a basis \( \{E, F, G\} \) in \( g \). So, we have to consider two new Lie brackets \( [G, E], [G, F] \in g' \), which we define to be \( [G, F] = aE + bF \) and \( [G, E] = aE + bF \supset [G, E] = aE + bF \) where \( a, b, c, d \in C \). Firstly

\[ g' = \text{span}\{[E, F], [G, E], [G, F]\} = \text{span}\{aE + bF, cE + dF\} \]

and the \( \dim g' = 2 \) thus we must have that the elements of the span \( aE + bF \), \( cE + dF \) are linearly independent. We could show this by saying,

\[ T = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0. \]

So it is seen that \( g \) is completely described by a \( 2 \times 2 \) non-singular matrix \( T \).

**FACT:** any such \( T \) does define a Lie algebra.

That is, we define a Lie bracket on \( g \) by \( [E, F] = 0 \), \( [G, E] = aE + bF \) and \( [G, F] = cE + dF \). We know the first two axioms of a Lie algebra hold for this
Lie bracket, all is left is to check the Jacobi identity for any $X, Y, Z \in \mathfrak{g}$. This is left, however, as an exercise.

We first pose a question. Can different matrices $T$ define the same Lie algebra (i.e. isomorphic)? The answer is yes!

Let us change the basis $\{E, F\}$ in $\mathfrak{g}'$ to another basis $\{\overline{E}, \overline{F}\}$, notice we still have $[E, F] = 0$. Then ($\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}$) is the matrix corresponding to the operator $\text{ad} G$ on $\mathfrak{g}'$. By definition of the adjoint operator we have that $(\text{ad} G)(E) = [G, E]$ and $(\text{ad} G)(F) = [G, F]$. If $C'$ is the coordinate change matrix from $\{E, F\}$ to $\{\overline{E}, \overline{F}\}$ then the “new” $T$ is $T' = \overline{T} = C^{-1}TC$. So by changing the basis $\{E, F\}$ in $\mathfrak{g}'$ we can replace $T'$ by $C^{-1}TC$ for all $C$.

Also I can choose “another $G$”, say $\overline{G}$, such that $\overline{G} = tG + X$ for all $t \in \mathbb{C}$ ($t \neq 0$) and $X \in \mathfrak{g}'$. This “new” $G$ gives the following bracket relationships with $E$ and $F$

$$[\overline{G}, E] = [tG + X, E] = t[G, E] = t(aE + bF)$$

So, this corresponds to $T$ being multiplied by any $t \neq 0$.

**Conclusion:** two different matrices, $T$ and $\overline{T}$, define the same $\mathfrak{g}$ iff $\overline{T}$ can be obtained from $T$ by a sequence of operations. i.e. $T \mapsto iT$, $t \neq 0$ and $T \mapsto (C')T(C')^{-1}$

(e) There is only one remaining case to consider, which is $\dim \mathfrak{g}' = 3$. So $(\mathfrak{g}' \subseteq \mathfrak{g} \mapsto \mathfrak{g} = \mathfrak{g}')$.

**Theorem 3.** If $\dim \mathfrak{g} = \dim \mathfrak{g}' = 3$ then $\mathfrak{g} \cong \mathfrak{sl}_2 \mathbb{C}$.

**Proof.** The plan is that we will choose a basis $\{H, E, F\}$ in $\mathfrak{g}$ such that $[H, E] = 2E$, $[H, F] = -2F$ and $[E, F] = H$. Then we can define a linear map $\varphi : \mathfrak{g} \rightarrow \mathfrak{sl}_2 \mathbb{C}$ by

$$H \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$E \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$F \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

They’re all in $\mathfrak{sl}_2 \mathbb{C} \Rightarrow$ a bijection. Why $\varphi$ is a Lie algebra homomorphism? Need to show
\[ \varphi([X, Y]) = [\varphi(X), \varphi(Y)]_{sl_C} = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X) \]

Because of linearity of \( \varphi \) it is enough to check \((*)\) only for \( X, Y \in \{ E, H, F \} \)

\[ 0 = \varphi([E, E]) = \varphi(E)\varphi(E) - \varphi(E)\varphi(E) = 0 \]

If I know that \((*)\) is OK for \( X = E, Y = F \) then it is OK for \( X = F, Y = E \) as well,

\[ \varphi([F, E]) = \varphi(F)\varphi(E) - \varphi(E)\varphi(F) \quad \text{by antisymmetry of } [\ , ]_g \]

\[ -\varphi([E, F]) = -\varphi(E)\varphi(F) + \varphi(F)\varphi(E) \]

Enough to check \((*)\) for \((X, Y) = (E, F), (E, H), (F, H)\). Then \((*)\) can be checked directly. \(\square\)

**Remark.**

(i) For all \( X \in g, X \neq 0 \) we have \( \text{rank}(\text{ad}X) = \dim(\text{Im}(\text{ad}X)) = 2 \). Complete \( X \) to a basis \( \{X, Y, Z\} \). Then \( g' = \text{span}\{[X, Y], [X, Z], [Y, Z]\} \). All the Lie brackets are linearly independent in \( g \). So,

\[ \text{Im}(\text{ad}X) = \text{span}\{\text{ad}X(X), \text{ad}X(Y), \text{ad}X(Z)\} \]
\[ = \text{span}\{[X, X], [X, Y], [X, Z]\} \]
\[ = \text{span}\{[X, Y], [X, Z]\} \]
\[ (= V) \]

So \( \dim(\text{Im(ad}X)) = 2 = \text{rank(ad}X) \).

(ii) \( \dim(\text{Im(ad}X)) + \dim(\ker(\text{ad}X)) = 2 + 1 = \dim g = 3 \). \( X \in \ker(\text{ad}(X)) = \mathbb{C}X = \text{span}\{X\}, \text{ad}(X)(X) = [X, X] = 0 \). Take any \( X \in g, X \neq 0 \). Consider \( \text{ad}(X) : g \to g \).

(A) Suppose \( \text{ad}(X) \) has at least one non-zero eigenvalue \( a \in \mathbb{C}, a \neq 0 \).

Let \( E \) be the corresponding eigenvector. Then

\[ [X, E] = -aE \]
\[ \frac{2X}{a}, E] = 2E \]

Let \( \frac{2X}{a} = H \) by definition, then \( [H, E] = 2E \).

(B) Now suppose that \( X \) does not have a non-zero eigenvalue \( \Leftrightarrow \) all eigenvalues of \( \text{ad}(X) \) are 0.

**FACT:** From the theory of matrices we know. Suppose \( T : V \to V \) (\( \text{ad}X : g \to g \)), \( \dim V = 3 \) such that all eigenvalues of \( T \) are 0 and
rank $T = 2$, rank$(\text{ad} X) = 2$. Then there exists a basis $\{v_1, v_2, v_3\}$ such that the matrix of $T$ relative to this basis is

$$
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
$$

Consequence of Jordan normal form of any $n \times n$ matrix. Now $T = \text{ad}(X)$, $v_1 \in \ker T$, $T(v_1) = 0$, $T(v_2) = v_1$. So $\text{ad}(X)(X) = [X, X] = 0$ ⇒ $v_1 = X$. Let $g \ni Y = v_2$. Then $[X, Y] = \text{ad}(X)(Y) = X = v_1$. So $[X, Y] = X$.

Put $H = -2Y$ and $E = X$. Then $[H, E] = [-2Y, X] = (-2)(-X) = 2E$. We have $[H, E] = 2E$ for some $H, E \in g$. Then $\text{ad}(H) : g \to g$. We know $H \in g = g' \Rightarrow H$ is a linear combination of brackets $[A, B]$ where $A, B \in g$. Also $\text{ad}(H)$ is a linear combination of $\text{ad}([A, B]) = [\text{ad}(A), \text{ad}(B)] = \text{ad}(A) \text{ad}(B) - \text{ad}(B) \text{ad}(A)$.

trace$(\text{ad}(H))$ is a linear combination of trace$(\text{ad}([A, B]))$ which is trace$(\text{ad}(A) \text{ad}(B)) - \text{trace}(\text{ad}(B) \text{ad}(A)) = 0$. Eigenvectors of $\text{ad} H$ such that $[H, H] = 0 = 0 \cdot H$ and $\text{ad}(H)(E) = [H, E] = 2E$. So $H$ has eigenvalue 0 and $E$ has eigenvalue 2. This implies there exists $F$ with eigenvalue -2 (as sum of eigenvalues must equal 0). $[H, F] = -2F$.

Need to get the third relation. Now,

$$
[H, [E, F]] + [E, [F, H]] + [F, [H, E]] = 0
$$

the Jacobi identity. So rearranging gives,

$$
[H, [E, F]] = [E, [H, F]] + [[H, F], F]
= [E, -2F] + [2E, F]
= 0
$$

using the relations $[H, E] = 2E$ and $[H, F] = -2F$. So $[E, F]$ an eigenvector for $ad H$ with zero eigenvalue $(\text{ad} H)([E, F]) = 0 \cdot [E, F] \Rightarrow [E, F] = cH$ for some $c \in \mathbb{C}$. Can $c = 0$? No, otherwise $[E, F] = 0$ and $F \in \ker(\text{ad}(E))$ but $\ker(\text{ad}(E)) = \mathbb{C} \cdot E$ (remark at the beginning of the proof). $[E, F] = cH$ for some $c \neq 0 \Leftrightarrow [\frac{E}{c}, F] = H$. Set $\tilde{E} = \frac{E}{c}$ then $[\tilde{E}, F] = H$. 

the relation $[H, E] = 2E$ still holds for $\hat{E} = \frac{E}{c}$. i.e. $[H, \hat{E}] = 2\hat{E}$ and now we have $[\hat{E}, E] = H$, the final relation required. (Note $[H, F] = -2F$ also holds for $\hat{E}$). End proof.

3. “Solvable” Lie Algebras

Remark. Differential equations $\sim$ Lie algebras and “solvable D.E.” $\sim$ solvable Lie algebras.

3.1. Quotient (of factor) vector space

$V$ - any vector space and $V \supseteq U$ a vector subspace. As a set

$$V/U = \{ \frac{x+U}{x'} : x \in V \}$$

“cosets” of the Abelian group $V$ with respect to $U$. We define the operation as;

$$(x_1 + U) + (x_2 + U) \overset{\text{def}}{=} (x_1 + x_2) + U$$

Suppose $x'_1$ such that $x_1 + U = x'_1 + U$ (1) and $x'_2$ such that $x_2 + U = x'_2 + U$ (2). Then,

$$(x'_1 + U) + (x'_2 + U) \overset{\text{def}}{=} (x'_1 + x'_2 + U)$$

$x_1 - x'_1 \in U$ by (1) and $x_2 - x'_2 \in U$ by (2). Adding these two statements together gives us $x_1 - x'_1 + x_2 - x'_2 \in U \Rightarrow x_1 + x_2 + U = x'_1 = x'_2 + U$. Similarly $\alpha \in \mathbb{C}$ and $x + U \in V/U$ such that

$$\alpha(x + U) \overset{\text{def}}{=} \alpha x + U$$

if $x + U = x' + U$ then

$$\alpha(x' + U) \overset{\text{def}}{=} \alpha x' + U$$

$x - x' \in U \Rightarrow \alpha(x - x') \in U \Rightarrow \alpha X + U = \alpha x' + U$.

Exercise: Check that these operations of vector addition and scalar multiplication satisfy all the axioms of a vector space for $V/U$. In fact, they will follow from the respective axioms for $V$ itself.

“0” in $V/U$ is the class (coset) $0 + U = u + U$ for all $u \in U$ where 0 is in $V$.

Definition: Let $\mathfrak{g}$ be any Lie algebra over $\mathbb{F}$ ($= \mathbb{C}$). Let $\mathfrak{h} \subseteq \mathfrak{g}$ be an ideal of $\mathfrak{g}$. The quotient Lie algebra $\mathfrak{g}/\mathfrak{h}$ is $\mathfrak{g}/\mathfrak{h}$ as a vector space as defined above.

Take $X + \mathfrak{h}, Y + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$ where $X, Y \in \mathfrak{g}$. Then,

$$[X + \mathfrak{h}, Y + \mathfrak{h}ℑ/ℑ]_{\mathfrak{g}/\mathfrak{h}} \overset{\text{def}}{=} [X, Y]_{\mathfrak{g}} + \mathfrak{h}$$

Suppose $X + \mathfrak{h} = X' + \mathfrak{h}$ (3) and $Y + \mathfrak{h} = Y' + \mathfrak{h}$ (4) for some $X', Y' \in \mathfrak{g}$. Then,
\[ [X' + h, Y' + h]_{g/h} = [X', Y']_g + h \]

Now (3) \(\Rightarrow X - X' = A \in h\) for \(X = X' + A\) and (4) \(\Rightarrow Y - Y' = B \in h\) with \(Y = Y' + B\).

Now,
\[ [X + h, Y + h]_{g/h} = [X, Y]_g + h = [X', Y'] + [X', B] + [A, Y'] + [A, B] + h = [X', Y'] + h \]
as required. Need to check that the definition of a Lie bracket on \(g/h\) satisfies the three axioms of a Lie algebra.

Claim: they follow from the respective axioms for \(g\) itself. Check this for revision.

Example 3.1. Take any Lie algebra \(g\) such that \(h = g' \subseteq g\) is an ideal of \(g\). Now \(g/g' \ni X + g', Y + g'\). Now,
\[ [X + g', Y + g']_{g/g'} = [X, Y]_g + g' = 0 + g' \]
the zero coset. Hence \(g/g'\) is Abelian.

Let \(g\) be any Lie algebra over \(F(= \mathbb{C})\). Then,

Definition: the derived series of \(g\) is \(g^0 = g \supset g^{(1)} = g' \supset g^{(2)} = (g')' \supset \ldots\)

Definition: \(g\) is solvable, if there exists \(p\) such that \(g^{(p)} = \{0\}\) so that the derived series "terminates".

Remark. \(g^{(p+1)} = g^{(p+2)} = \ldots = \{0\}\)

Example 3.2. \(g = \{X = [x_{ij}]_{i,j=1}^n \in \text{Mat}_n \mathbb{C} | x_{ij} = 0 \text{ if } i \geq j\}\). Typical \(X\) may be,
\[
\begin{pmatrix}
0 & * & * \\
0 & 0 & * \\
0 & 0 & 0
\end{pmatrix}
\]
a basis of \(g\) is \(\{E_{ij} | i < j\}\) where \([E_{ij}, E_{kl}] = \delta_{ij}E_{il} - \delta_{li}E_{kj}\) for all \(i, j, k, l = 1, \ldots, n\).

Lemma 1. \(g^{(p)} \subseteq \text{span}\{E_{ij} : j - i \geq 2^p\}\) (*)

Proof. We prove this by induction
(1) \(p = 0, g^{(p)} = g\) where \(g = \text{span}\{E_{ij} | i < j\}\), Now, \(i < j \Leftrightarrow j - i > 0 \Leftrightarrow j - i \geq 1 = 2^0\).
(2) Induction step: assume \((\ast)\) is true. Then,

\[
g^{(p+1)} = (g^{(p)})' = \text{span}\{[X,Y], X, Y \in g^{(p)}\} \\
\subseteq \text{span}\{[E_{ij}, E_{kl}] | j - i \geq 2^p, l - k \geq 2^p\} \\
= \text{span}\{\delta_{jk}E_{il} - \delta_{li}E_{kj} | j - i, l - k \geq 2^p\}
\]

\[j - i, l - k \geq 2^p \Rightarrow j - i + l - k \geq 2, \text{ where } 2^p = 2^{p+1} \]

\[\subseteq \{E_{ab} | b - a \geq 2^{p+1}\}\]

\[\square\]

Remark. In fact, we have equality instead of \(\subseteq\) in \((\ast)\) (need to prove \(\supseteq\) in \((\ast)\)). This is an exercise.

The Lemma implies that \(g\) is solvable. Indeed, suppose \(p\) is so large that \(n < 2^p\). Then there is no pair \((i,j)\) with \(i, j = 1, \ldots, n\) such that \(j - i \geq 2^p\) \((j - i < n\) anyway). Then, \(g^{(p)} \supset \text{span()} = \{0\}\).

**Example 3.3.** \(g = gl_n \mathbb{C}, n > 1\) \((g = gl_1 (\mathbb{C})\)-abelian, \(g' = \{0\} \Rightarrow \text{solvable}\). Now \(n > 1\). Then

\[(gl_n \mathbb{C})' = sl_n \mathbb{C}\]

\[((sl_n \mathbb{C})')' = sl_n \mathbb{C}\]

\(g^{(0)} = gl_n \mathbb{C}, g^{(1)} = sl_n \mathbb{C}; g^{(1)} = g^{(2)} = \cdots = sl_n \mathbb{C} \neq \{0\}\). The derived series does NOT terminate and therefore \(g\) is not solvable.

### 3.2. Theorems on solvable Lie algebras

A) Let \(g\) be solvable and \(g \supseteq h\) a subalgebra. Then \(h\) is solvable as well.

B) Let \(\varphi : g \rightarrow h\) be a Lie algebra homomorphism onto \(h\). Let \(g\) be solvable. Then \(h\) is solvable as well.

C) \(g \supseteq h\) an ideal of \(g\). Suppose that \(g/h\) is a solvable Lie algebra and that \(h\) is solvable. Then \(g\) is solvable as well.

**Proof.** A) Now we have,
\[ g^{(0)} = g \supseteq h = h^{(0)} \]
\[ g' \supseteq h' \]
\[ g'' \supseteq h'' \]
\[ \vdots \]
\[ g^{(p)} \supseteq h^{(p)} \]

\( p \) large \( \Rightarrow \) \( g^{(p)} = \{0\} \) \( \Rightarrow \) \( h^{(p)} = \{0\} \).

\[ h' = \text{span}\{[X, Y]|X, Y \in h\} = \text{span}\{[\varphi(\tilde{X}), \varphi(\tilde{Y})]|\tilde{X}, \tilde{Y} \in g\} = \text{span}\{\varphi[\tilde{X}, \tilde{Y}]|\tilde{X}, \tilde{Y} \in g\} = \varphi(g') \]

So \( h'' = \cdots = \varphi(g'') \) and so on \( h^{(p)} = \varphi(g^{(p)}) \) for all \( p = 0, 1, 2, \ldots \) there exists \( p \) such that \( g = \{0\} \Rightarrow h^{(p)} = \varphi(\{0\}) = \{0\} \).

**Proof.**

C) \( g/h \) solvable. There exists \( p \) such that \( (g/h)^{(p)} = \{0\} \). Consider the *canonical homomorphism*

\[ \pi : g \rightarrow g/h \]
\[ X \mapsto X + h \]

(Linear map by definition of the vector space \( g/h \)). Homomorphism property of \( \pi \),

\[ \pi([X, Y]) = [X, Y] + h = [X + h, Y + h] = [\pi(X), \pi(Y)]_{g/h} \]

\( \pi \) a homomorphism \( \Rightarrow \) arguments of B. Show that \( \pi(g^{(p)}) = \pi(g)^{(p)} \) (\( \varphi = \pi \) (only the homomorphism property of \( \varphi = \pi \) is required for this). Now \( \pi(g^{(p)}) = (g/h)^{(p)} = \{0\} \)

because \( \pi \) is onto by definition for some \( p \). Now,

\[ \pi(g^{(p)}) = \{0\} \Leftrightarrow g^{(p)} \subseteq \ker \pi \]

\( \ker \pi = \{X \in g|X + h = h\} = \{X \in h\} = h \). So \( g^{(p)} \subseteq h \) for \( p \) large enough \( h \) is solvable \( \Rightarrow \) there exists \( q \) such that \( h^{(q)} = \{0\} \).

Then,

\[ g^{(p+q)} \supseteq h^{(q)} = \{0\} \Rightarrow g^{(p+q)} = \{0\} \]

which implies \( g \) is solvable. \( \square \)
Lie Theorem (on solvable Lie algebras)

Let $V$ be any finite dimensional vector space, over $\mathbb{C}$. Consider the general linear Lie algebra of $V$, $\mathfrak{gl}(V) = \{X : V \to V \text{ linear}\}$; $[X, Y] = XY - YX$. We know that $\mathfrak{gl}(V) \cong \mathfrak{gl}_n(\mathbb{C})$ where $n = \dim(V)$, so $\mathfrak{gl}(V)$ NOT solvable for $n > 1$.

Let $\mathfrak{gl}(V) \supseteq \mathfrak{g}$ be a Lie subalgebra which is SOVABLE.

**Theorem 4.** then $V$ contains a vector $v$, which is an eigenvector for all $X \in \mathfrak{g}$; that is $Xv = \lambda(X)v$ for all $X \in \mathfrak{g}$.

**Comment:** take any set $S \subseteq \mathfrak{gl}(V)$. Suppose we look for a common eigenvector for all $X \in S$. Take a Lie subalgebra $\mathfrak{gl}(V) \supseteq \mathfrak{g}$ spanned by $S$ (i.e. $\mathfrak{g}$ closed with respect to $[X, Y] = XY - YX$). Then if $\mathfrak{g}$ is solvable then there exists $v$.

**Corollary 1** (to Lie Theorem). there is a “suitable” basis of $V$ such that the matrices of all $X \in \mathfrak{g}$ relative to this basis are upper triangular. For example,

$$
X = \begin{pmatrix}
\lambda & * & * & * \\
0 & \ddots & \ddots & * \\
0 & \ddots & \ddots & * \\
0 & 0 & 0 & *
\end{pmatrix},
v = \begin{pmatrix}
\lambda \\
0 \\
0 \\
0
\end{pmatrix}, \quad Xv = \lambda v
$$

**Remark.** The corollary implies the Theorem by the rules of matrix presentation of $X \in \mathfrak{g}$.

**Proof.** By induction on $\dim V = n$. If $\dim V = 0$. Then $V = \{0\}$ and $\mathfrak{gl}(V) = 0$. So the statement to prove is trivial. Now suppose that the Corollary is true for all vector spaces $U$ (instead of $V$) such that the dimension of $U < n = \dim V$. (On the other hand, by the Lie theorem itself there exists $v_1 \neq 0 \in V$ such that $Xv_1 = \lambda(X)v_1$ for all $X \in \mathfrak{g}$, $\dim \mathbb{C}v_1 = 1$).

Consider the quotient space $U = V/\mathbb{C}v_1$, now we have $\dim U = \dim(V) - 1 = n - 1$. So induction assumption applies to $U$. Consider the map

$$
\varphi : \mathfrak{g} \to \mathfrak{gl}(U), \quad X \mapsto \overline{X}
$$

where

$$
\overline{X}(v + \mathbb{C}v_1) = Xv + \mathbb{C}v_1.
$$

This map $\varphi$ is linear. Moreover, this mapping $\varphi$ is a Lie algebra homomorphism. That is, need to check that $\varphi([X, Y]_{\mathfrak{g}}) = [\varphi(X), \varphi(Y)]_{\mathfrak{gl}(U)}$, where we have $\varphi([X, Y]_{\mathfrak{g}}) = \varphi(XY - YX) = \overline{XY} - \overline{YX}$ and by definition $[\varphi(X), \varphi(Y)]_{\mathfrak{gl}(U)} = \overline{XY} - \overline{YX}$. Apply the LHS and RHS to any $v + \mathbb{C}v_1$. 
\[
\overline{XY}(v + C v_1) = XYv + C v_1 \\
\overline{X} \cdot \overline{Y}(v + C v_1) = \overline{X}(Yv + C v_1) = XYv + C v_1
\]
similarly \[\overline{Y}X(v + C v_1) = \overline{Y} \cdot \overline{X}v + C v_1.\]

So homomorphism property is OK for \(\varphi\). We need to check that \(\varphi\) is defined correctly. Suppose \(v + C v_1 = v' + C v_1\) \((\ast)\). Then,
\[
\overline{X}(v + C v_1) = Xv + C v_1 \\
Xv' + C v_1 = \overline{X}(v' + C v_1)
\]
Need to show that \(Xv + C v_1 = Xv' + C v_1\) where \(X(v - v') \in C v_1 \Rightarrow X(C v_1) \subset C v_1 \Rightarrow X(v - v') \in C v_1\) because \(v_1\) is an eigenvector for \(X\).

Now we can use Theorem B on solvability. The image of \(\varphi\) in \(gl(U)\) is a solvable Lie subalgebra in \(gl(U)\). By the induction assumption, there is a “good” basis \(u_2, \ldots, u_n\) of \(U\) such that all elements of \(\varphi(g) \subset gl(U)\) get represented by upper triangular matrices.

\[
\varphi(g) \ni \varphi(X) = \overline{X}. \text{ Then,}
\]
\[
(\dagger) \quad \varphi(X)u_j = \sum_{i=2}^{n} a_{ij}u_j \quad j = 2, \ldots, n
\]
u_j = v_j + C v_1 for \(j = 2, \ldots, n\). Now the left hand side of \((\dagger)\) is,
\[
\overline{X}(u_j) = \overline{X}(v_j + C v_1) = Xv_j + C v_1 = \sum_{i=2}^{n} a_{ij}(v_i + C v_1)_{i \leq j}
\]
which implies \(Xv_j \in C v_1 \oplus \cdots \oplus C v_j\). Then,
\[
(\ast\ast) \quad Xv_j = \sum_{i=1}^{j} b_{ij}v_i \quad \text{for all } j = 2, \ldots, n
\]
Also \(Xv_1 = \lambda v_2\), so \((\ast\ast)\) also holds for \(j \equiv 1\). So the matrix of \(X\) is upper triangular relative to the basis of \(V\), \(\{v_1, v_2, \ldots, v_n\}\) where \(v_2, \ldots, v_n\) are representatives of \(u_2, \ldots, u_n\).

\[\square\]

**Lemma 2.** Let \(\varphi : g \rightarrow h\) a Lie algebra homomorphism and \(\mathfrak{f} \subseteq h\) an ideal. Then \(\varphi^{-1}(\mathfrak{f}) \subseteq g\) is an ideal of \(g\).

**Proof.** \(\varphi^{-1}(\mathfrak{f}) = \{X \in g|\varphi(X) \in \mathfrak{f}\}\). Take any \(X \in \varphi^{-1}(\mathfrak{f}), Y \in g\). We want to prove that
\[
[X, Y] \in \varphi^{-1}(\mathfrak{f}) \iff \varphi([X, Y]) \in \mathfrak{f} \iff [\varphi(X), \varphi(Y)] = \varphi([X, Y]) \in \mathfrak{f}
\]
but this is true because \(\mathfrak{f}\) is an ideal.
Remark. $\varphi^{-1}(f)$ is a subspace of $g \leq \varphi$.

**Lemma 3** (Dynkin’s Lemma). Let $V$ be a vector space, $\dim V < \infty$ and $\mathfrak{gl}(V) \supset g$ any subalgebra. Let $\mathfrak{h} \subseteq g$ be any ideal and let $\lambda : \mathfrak{h} \to \mathbb{C}$ be a linear function. Consider $W = \{v \in V | Yv = \lambda(Y)v \forall Y \in \mathfrak{h}\}$, note $0 \in W \Rightarrow W \neq \emptyset$ and also $W$ is a subspace, i.e. $W \subseteq V$. Then $W$ is $g$-invariant, that is for all $v \in W$, $X \in g$ then $Xv \in W$.

**Proof.** later but non-examinable.

**Proof.** (of Lie Theorem) By induction on $\dim g$. $g \subseteq \mathfrak{gl}(V) \Rightarrow \dim g \leq \dim \mathfrak{gl}(V) = (\dim V)^2 < \infty$. Case $\dim g = 0$, the induction base. Then $g = \{0\}$, $0.v = 0v$. So any non-zero $v \in V$ is an eigenvector.

Assume now that $\dim g > 0$ and that the Lie Theorem holds for Lie algebras of $\dim < \dim g$.

Three steps now,

1. $g$ is solvable $\Rightarrow g \neq g'$. Otherwise the derived series of $g$ would be $g^{(0)} = g$, $g^{(1)} = g' = g$, $g, g' \neq \{0\}$. So would not terminate. So $g' \subseteq g, g' \neq g \Rightarrow g/g' \neq \{0\}$ and $\dim g/g' > 1$. Take any subspace $f \subseteq g/g'$ such that $\dim f = \dim(g/g') - 1 > 0$.

   We know that $g/g'$ is an Abelian Lie algebra, which implies that $f$ is an ideal of $g/g'$ for all $X \in g/g'$, $Y \in f$ we have $[X, Y] = 0 \in f$.

   Now apply Lemma 1 to $g = g, \mathfrak{h} = g/g'$ and $f = f$. $\varphi = \pi : g \to g/g'; X \mapsto X + g'$ a canonical homomorphism. Then $\pi^{-1}(f) \subseteq g$ is an ideal by Lemma 1. So we have $\pi^{-1}(f)$ is solvable by Theorem A on solvability and $g$ solvable. We have,

   $$\pi^{-1}(f) \subseteq g \subseteq \mathfrak{gl}(V)$$

   and

   $$\dim \pi^{-1}(f) = \dim(g/g') - 1 + \dim(\ker \pi)$$

   $$= (\dim g - \dim g') - 1 + \dim g'$$

   $$= \dim g - 1$$

   $$< \dim g$$

2. Induction assumption applies to $\pi^{-1}(f) \subseteq \mathfrak{gl}(V)$. Then there exists a $\tilde{v} \in V$ with $\tilde{v} \neq 0$ such that $X\tilde{v} = \lambda(X)\tilde{v}$ for all $X \in \pi^{-1}(f)$. Here $\lambda : \pi^{-1}(f) \to \mathbb{C}$ is a linear function. So, consider $W = \{v \in V | Xv = \lambda(X)v \forall X \in \pi^{-1}(f)\} \neq \emptyset$ because $\tilde{v} \in W$. Apply Dynkin Lemma to $\lambda$ and $\mathfrak{h} = \pi^{-1}(f)$. Then this $W$ is $g$-invariant by Dynkin Lemma.

3. $\dim \pi^{-1}(f) = \dim g - 1$. Write $g = \pi^{-1}(f) \oplus \mathbb{C}X_0$ ($X_0$ is a complementary vector to $\pi^{-1}(f)$, i.e. $X_0 \not\in \pi^{-1}(f)$, by definition of direct sum) for some $X_0 \in g$ (this is a direct sum of vector spaces, not necessarily of Lie algebras). So, $X_0v \in W$ for all $v \in W$. So $X_0|_W : W \to W$ a linear transformation. Because $V$ is a vector space over $\mathbb{C}$ so is $W$ and $\dim W > 0$, $X_0|_W$ is a linear transformation of $W$. Hence there is an eigenvector $v_0 \in W$ for $X_0|_W$. So $X_0v_0 = \mu.v_0, v_0 \neq 0$. Claim this $v_0$ is an
eigenvector for all \( X \in \mathfrak{g} \). Take any \( X \in \mathfrak{g} = \pi^{-1}(f) \oplus \mathbb{C} X_0 \). So \( X = Y + zX_0 \), for \( z \in \mathbb{C} \) and \( Y \in \pi^{-1}(f) \). So,

\[
Xv_0 = (Y + zX_0)v_0 = Yv_0 + zX_0v_0 = \lambda(Y)v_0 + \mu zv_0 = (\lambda(Y) + \mu z)v_0
\]

Remark.

(1) By Lie theorem (by its Corollary) any solvable subalgebra of \( \mathfrak{gl}(V) \) can be represented by upper triangular matrices (choosing a good basis in \( V \)).

(2) Also, take any basis \( \{v_1, \ldots, v_n\} \) in \( V \) and identify \( \mathfrak{gl}(\mathbb{C}) = \mathfrak{gl}_n(\mathbb{C}) \). Consider \( u = \{A \in \mathfrak{gl}_n(\mathbb{C})|A_{ij} = 0 \text{ for } i > j\} \) (Exercise: check that \( u \) is solvable). Now, take any \( \mathfrak{g} \subset u \) a subalgebra. By Theorem A, \( \mathfrak{g} \) is solvable.

(3) This means that \( \mathfrak{g} \subseteq \mathfrak{gl}(V) \) is solvable iff \( \mathfrak{g} \) can be represented by upper triangular matrices.

4. Semi-simple Lie Algebras

Let \( \mathfrak{g} \) be any finite dimensional Lie algebra over \( \mathbb{C} \).

Lemma 4. Let \( \mathfrak{h}_1, \mathfrak{h}_2 \subseteq \mathfrak{g} \) be any ideals of \( \mathfrak{g} \). Then,

\[
\frac{\mathfrak{h}_1 + \mathfrak{h}_2}{\mathfrak{h}_2} \cong \frac{\mathfrak{h}_1}{\mathfrak{h}_1 \cap \mathfrak{h}_2}.
\]

Proof. Define a map \( \varphi : \frac{\mathfrak{h}_1 + \mathfrak{h}_2}{\mathfrak{h}_2} \rightarrow \frac{\mathfrak{h}_1}{\mathfrak{h}_1 \cap \mathfrak{h}_2} \) as follows; take any

\[
X_1 + (X_2 + \mathfrak{h}_2) \in \frac{\mathfrak{h}_1 + \mathfrak{h}_2}{\mathfrak{h}_2}
\]

N.B. \( X_2 + \mathfrak{h}_2 = X_1 + \mathfrak{h}_2 \mapsto X_1 + (\mathfrak{h}_1 \cap \mathfrak{h}_2) \). Then \( \varphi \) is linear and onto. To show it is a bijection we only need to show it is 1-1. So, if

\[
\varphi(X_1 + \mathfrak{h}_2) = \varphi(X_1' + \mathfrak{h}_2)
\]

for some \( X_1, X_1' \in \mathfrak{h}_2 \). Then,

\[
X_1 + (\mathfrak{h}_1 \cap \mathfrak{h}_2) = X_1' + (\mathfrak{h}_1 \cap \mathfrak{h}_2) \quad \text{or} \quad X_1 - X_1' \in \mathfrak{h}_1 \cap \mathfrak{h}_2.
\]

\( \Rightarrow X_1 - X_1' \in \mathfrak{h}_2 \) in particular. Then \( X_1 + \mathfrak{h}_2 = X_1' + \mathfrak{h}_2 \) and hence \( \varphi \) is 1-1. Now we need to show that \( \varphi \) is a homomorphism. So, take any \( X_1, Y_1 \in \mathfrak{h}_1 \) so \( X_1 + \mathfrak{h}_2, Y_1 + \mathfrak{h}_2 \in \frac{\mathfrak{h}_1 + \mathfrak{h}_2}{\mathfrak{h}_2} \).

Their Lie bracket is then,

\[
[X_1 + \mathfrak{h}_2, Y_1 + \mathfrak{h}_2] = [X_1, Y_1] + \mathfrak{h}_2
\]
\[ [X_1, Y_1] \mapsto [X_1, Y_1] + (h_1 + h_2) \in \frac{h_1}{h_1 \cap h_2} \]

\[ \varphi(X_1 + h_2), \varphi(Y_1 + h_2) = [X_1 + h_1 \cap h_2, Y_1 + h_1 \cap h_2] \]

\[ = [X_1, Y_1]_\varphi + (h_1 \cap h_2) \]

Hence \([X_1 + h_2, Y_1 + h_2] = [\varphi(X_1 + h_2), \varphi(Y_1 + h_2)]\) and so \(\varphi\) is a homomorphism. \(\square\)

**More Theorems on solvable Lie Algebras**

D) Let \(h_1, h_2 \subseteq g\) be any solvable ideals of \(g\). Then \(h_1 + h_2 = \{X_1 + X_2 \in g | X_1 \in h_1, X_2 \in h_2\}\) is also a solvable ideal of \(g\).

**Proof.** Let \(Y \in g\). Then, examining the Lie bracket we see

\[ [X_1 + X_2, Y] = [X_1, Y] + [X_2, Y] \in h_1 + h_2. \]

Now proving the solvability of \(h_1 + h_2\); consider

\[ \pi : h_1 \to \frac{h_1}{h_1 \cap h_2} \]

\[ X_1 \mapsto X_1 + (h_1 \cap h_2) \]

Theorem B (of solvability) implies that \(\frac{h_1}{h_1 \cap h_2}\) is solvable because \(h_1\) is solvable, \(h_1 \cap h_2 \subseteq h_2\).

By Theorem A we have that \(h_1 \cap h_2\) is solvable. Note, \(h_1\) is solvable by assumption. Next step is to use the Lemma;

\[ \frac{h_1}{h_1 \cap h_2} \cong \frac{h_1 + h_2}{h_2} \]

\[ \text{solvable } \Rightarrow \text{solvable.} \]

Now,

\[ \left\{ \begin{array}{c}
\frac{h_1 + h_2}{h_2} \text{ is solvable} \\
\frac{h_1}{h_2} \text{ is solvable}
\end{array} \right\} \Rightarrow \text{Thm C } h_1 + h_2 \text{ is solvable.} \]

\(\square\)

**Definition:** The **Radical** of any Lie algebra \(g\), is the maximum solvable Lie algebra of \(g\), \(\mathcal{R}(g)\). That is, if \(h \subseteq g\) is any solvable ideal, then

\[ \mathcal{R}(g) \supseteq h \]

**Proposition.** \(\mathcal{R}(g)\) exists for any finite dimensional Lie algebra \(g\).
Proof. Let \( \{ h_\alpha : \alpha \in A \} = H \) be all solvable ideals of \( g \). Pick up \( h = h_1 \in H \). Then, take another, say \( h_2 \in H \). Consider \( h_1 + h_2 \in H \). Take another, \( h_3 \in H \), then, \( (h_1 + h_2) + h_3 \in H \) and so on ...

At each step, there are 2 possibilities either:

\[
(h_1 + \cdots + h_{k-1}) + h_k \neq (h_1 + \cdots + h_{k-1})
\]

or

\[
(h_1 + \cdots + h_{k-1}) + h_k = h_1 \cdots h_{k-1} \quad \text{obsolete step.}
\]

Therefore no need to make steps of the second kind. So, assume,

\[
h_1 \subset h_1 + h_2 \subset h_1 + h_2 + h_3 \cdots \subseteq g
\]

\[
\dim d_1 < d_2 < d_3 < \cdots < d = \dim g.
\]

So process will terminate, can’t add anything else, then

\[
h_1 + \cdots + h_k = \mathcal{R}(g)
\]

\( \square \)

Definition: A Lie algebra \( g \) is semi-simple if \( \mathcal{R}(g) = \{0\} \). That is, the only solvable ideal of \( g \) is \( \{0\} \).

Theorem 5. \( \frac{\mathcal{R}(g)}{\mathcal{R}(g)} \) is always semi-simple.

Proof. Suppose \( \frac{\mathcal{R}(g)}{\mathcal{R}(g)} \) is not semi-simple, so there exists \( h \subseteq \frac{\mathcal{R}(g)}{\mathcal{R}(g)} \) a solvable ideal; \( h \neq \{0\} \). Canonical homomorphism,

\[
\pi : g \rightarrow \frac{g}{\mathcal{R}(g)}
\]

\[
\pi^{-1}(h) \mapsto h
\]

ideal \( \Leftarrow \) ideal

then

\[
\pi : \pi^{-1}(h) \rightarrow h
\]

\[
h \cong \frac{\pi^{-1}(g)}{\ker \pi}
\]

solvable

\[
= \frac{\pi^{-1}(h)}{\mathcal{R}(g)}
\]
solvable.

\( \mathcal{R}(\mathfrak{g}) \) is solvable. By Theorem C we have that \( \pi^{-1}(\mathfrak{h}) \) is solvable, \( \pi^{-1}(\mathfrak{h}) \subset \mathcal{R}(\mathfrak{g}) \) maximal.

\[
\pi(\pi^{-1}(\mathfrak{h})) = \{0\} \quad \text{not solvable.}
\]

\[
\pi(\mathcal{R}(\mathfrak{g})) = \{0\} \quad \square
\]

**Theorem 6** (on semi-simple Lie algebras). Suppose \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) are both semi-simple Lie algebras. Then \( \mathfrak{g}_1 \oplus \mathfrak{g}_2 \) is also semi-simple.

**Proof.** Consider \( \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \supseteq \mathfrak{h} \) - solvable ideal. Need to show that \( \mathfrak{h} = \{0\} \). So that \( \mathcal{R}(\mathfrak{g}) = \{0\} \). Consider \( \pi_1 : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}_1; (X_1, X_2) \mapsto X_1 \).

**Claim:** \( \pi_1(\mathfrak{h}) \) is an ideal of \( \mathfrak{g}_1 \) then \( \mathfrak{h} \)-ideal \( \Rightarrow \) for all \( (X_1, X_2) \in \mathfrak{h} \) and for all \( (Y_1, Y_2) \in \mathfrak{g}_1 \oplus \mathfrak{g}_2 = \mathfrak{g} \).

\[
\mathfrak{h} \supseteq [(X_1, X_2), (Y_1, Y_2)] = \left( [X_1, Y_1]_{\mathfrak{g}_1}, [X_2, Y_2]_{\mathfrak{g}_2} \right)
\]

\[
\pi_1(\mathfrak{h}) \supseteq [X_1, Y_1]_{\mathfrak{g}_1} \quad \text{for all } X_1 \in \pi_1(\mathfrak{h}) \text{ and for all } Y_1 \in \mathfrak{g}_1
\]

\( \pi_1(\mathfrak{h}) \) is a vector subspace of \( \mathfrak{g}_1 \) because \( \pi_1 \) is linear. So \( \pi_1(\mathfrak{h}) \) is an ideal. Next, \( \pi_1 \) is a homomorphism (Exercise). Then, by Theorem B on solvability, \( \pi_1(\mathfrak{h}) \subseteq \mathfrak{g}_1 \) is solvable. But \( \mathfrak{g}_1 \) is semi-simple \( \Rightarrow \) \( \pi_1(\mathfrak{h}) = \{0\} \) for all \( (X_1, X_2) \in \mathfrak{h}, X_1 = 0 \) (and we prove that \( X_2 = 0 \)). Similarly, using \( \pi_2 : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}_2; (X_1, X_2) \mapsto X_2 \Rightarrow \mathfrak{h} = \{0\} \quad \square
\]

**Definition:** \( \mathfrak{g} \) is simple, if \( \dim \mathfrak{g} > 1 \), and \( \mathfrak{g} \) has no ideals except for \( \mathfrak{g} \) itself and \( \{0\} \).

**Example 4.1.** \( \mathfrak{g} = \mathfrak{sl}_2 \mathbb{C} \) (Exercise 15, proof given already). \( \dim \mathfrak{g} = 3 > 1 \).

**Proposition.** \( \mathfrak{g}_1, \ldots, \mathfrak{g}_k \) are simple \( \Rightarrow \mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \) is semi-simple.

**Proof.** by induction on \( k \).

(1) \( k = 1 \). Suppose \( \mathfrak{g} = \mathfrak{g}_1 \) is simple. Need to show that \( \mathfrak{g} \) is semi-simple. We know that the only ideals of \( \mathfrak{g} \) are \( \mathfrak{g} \) and \( \{0\} \). Take any solvable ideal, \( \mathfrak{h} \), of \( \mathfrak{g} \). If \( \mathfrak{h} = \{0\} \) we are done. So, we are left with the case when \( \mathfrak{h} = \mathfrak{g} \). Can \( \mathfrak{g} = \mathfrak{h} \) be solvable? Consider \( \mathfrak{g} \supseteq \mathfrak{g}' \supseteq \mathfrak{g}'' \supseteq \ldots \)

Then \( \mathfrak{g}' \) is an ideal of \( \mathfrak{g} \). If \( \mathfrak{g}' = \{0\} \) then \( \mathfrak{g} \) is Abelian, \( \dim \mathfrak{g} > 1 \) because \( \mathfrak{g} \) is simple. Take \( \mathfrak{f} \subseteq \mathfrak{g} \) a subspace such that \( \dim \mathfrak{f} = 1 \). Then \( \mathfrak{f} \subseteq \mathfrak{g} \) is an ideal. So \( \mathfrak{f} \) ideal of \( \mathfrak{g} \) different from \( \mathfrak{g}, \{0\} \) but this is a contradiction. Hence \( \mathfrak{g}' \neq \{0\} \Rightarrow \mathfrak{g} = \mathfrak{g}' = \mathfrak{g}'' = \mathfrak{g}''' = \ldots \), does not terminate. \( \mathfrak{g} \) is not solvable. This completes the proof for \( k = 1 \).
(2) **Induction step.** $g_1 \ldots g_k$ are simple and let the induction assumption be $g_1 \oplus \cdots \oplus g_{k-1}$ is semi-simple.

$$g = (g_1 \oplus \cdots \oplus g_{k-1}) \oplus g_k$$

semi-simple by Theorem □

**Theorem 7** (without proof). Let $g$ be any semi-simple Lie algebra, finite dimensional over $\mathbb{C}$ (or $\mathbb{R}$). Then there exist ideals $g_1, \ldots, g_k$ of $g$ such that each of $g_1, \ldots, g_k$ is a simple Lie algebra and $g \cong g_1 \oplus \cdots \oplus g_k$. Moreover, these $g_1, \ldots, g_k$ are unique up to permuting them.

## 5. Classification of Simple Lie Algebras

**Theorem 8** (without proof). Any simple finite dimensional Lie algebra over $\mathbb{C}$ is isomorphic to one in the following table.

<table>
<thead>
<tr>
<th>Case</th>
<th>Dimension</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_\ell, \ell \geq 1$</td>
<td>$(\ell + 1)^2 - 1$</td>
<td>$\mathfrak{sl}_{\ell + 1}$ $\mathbb{C}$</td>
</tr>
<tr>
<td>$B_\ell, \ell \geq 1$</td>
<td>$\frac{3\ell(2\ell + 1)}{2}$</td>
<td>$\mathfrak{so}_{2\ell + 1}$ $\mathbb{C}$</td>
</tr>
<tr>
<td>$C_\ell, \ell \geq 1$</td>
<td>$\ell(2\ell + 1)$</td>
<td>$\mathfrak{so}_{2\ell}$ $\mathbb{C}$</td>
</tr>
<tr>
<td>$D_\ell, \ell \geq 3$</td>
<td>$\frac{3\ell(2\ell - 1)}{2}$</td>
<td>$\mathfrak{sp}_{2\ell}$ $\mathbb{C}$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>14</td>
<td>$\mathfrak{g}_2$</td>
</tr>
<tr>
<td>$F_4$</td>
<td>52</td>
<td>$\mathfrak{f}_4$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>78</td>
<td>$\mathfrak{e}_6$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>133</td>
<td>$\mathfrak{e}_7$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>248</td>
<td>$\mathfrak{e}_8$</td>
</tr>
</tbody>
</table>

**Comments**

(1) all algebras in the table are not isomorphic, except for $B_2 \cong C_2$; $D_3 \cong A_3$

(2) $D_1 = \mathfrak{so}_2 = \left\{ \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} \bigg| x \in \mathbb{C} \right\}$, Abelian with dim = 1.

$D_2 = \mathfrak{so}_4 \mathbb{C} = A_1 \oplus A_1 = \mathfrak{sl}_2 \mathbb{C} \oplus \mathfrak{sl}_2 \mathbb{C}$ (already mentioned.)

(3) About proof: any simple Lie algebra can be described by a Dynkin diagram, with $\ell$ vertices. Then classify diagrams.

**Lemma 5.** Let $\mathfrak{g}(V) \supseteq \mathfrak{g} \supseteq \mathfrak{h}$ be an ideal of a Lie algebra $\mathfrak{g}$ and let $\lambda : \mathfrak{h} \to \mathbb{C}$ be any linear function. Consider the subspace of $V$

$$W = W_\lambda = \{ v \in \mathfrak{g} | Yv = \lambda(Y)v \text{ for all } Y \in \mathfrak{h} \}.$$  

Then $W$ is $\mathfrak{g}$-invariant, i.e. $Xv \in W$ for all $X \in \mathfrak{g}, v \in W$.

**Example 5.1.** It may be that $W_\lambda = \{ 0 \}$. $0 = X \cdot 0 \in \{ 0 \}$.

**Proof.** (non exminable).

(1) Take any $X \in \mathfrak{g}, Y \in \mathfrak{h}$ and $v \in W = W_\lambda; v \neq 0$. Then $Yv = \lambda(Y)v$. Need to show that $XYv \in W$ that is $YXv = \lambda(Y)Xv \ (\ast)$.

$$YXv = (XY + YX - XY)v$$

$$= XYv + [Y, X]_v$$
and \( \mathfrak{h} \) is an ideal
\[
\mathfrak{h} = \lambda(Y)Xv + \lambda([Y,X])v
\]

(2) Let us prove that \( \lambda([Y,X]) = 0 \), then (\( \ast \)) OK. Fix \( X \in \mathfrak{g} \). Consider the vectors
\[
v_0 = v, \quad v_1 = Xv, \quad v_2 = X^2v, \ldots, \quad v_i = X^iv
\]
and consider \( U_i = \text{span}(v_0, v_1, \ldots, v_i) \)
\[
U_0 \subseteq U_1 \subseteq U_2 \subseteq \cdots \subseteq U_i \subseteq U_{i+1} \subseteq \cdots \subseteq V.
\]
Can it happen that \( U_i \subset U_{i+1} \), \( \dim U_i \leq \dim U_{i+1} \) for all \( i \)? No, because \( \dim U_0 \leq \dim U_1 \leq \cdots \leq \dim V < \infty \). So, let \( k = \min \{ i | U_i = U_{i+1} \} \) - exists.
\[
U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_k = U_{k+1} = U_{k+2} = U_{k+3} = \cdots \subseteq V
\]
For instance, let us check that \( U_{k+1} = U_{k+2} \). So,
\[
\begin{align*}
U_{k+2} &= \text{span}(v_0, v_1, \ldots, v_k, v_{k+2}) \\
U_{k+1} &= \text{span}(v_0, v_1, \ldots, v_{k+1})
\end{align*}
\]
We have that \( U_{k+1} \subseteq U_{k+2} \subseteq V \). Need to show that \( v_{k+2} = X^{k+2}v \in U_{k+1} \). We know that
\[
\begin{align*}
U_{k+1} &= U_k = \text{span}(v_0, v_1, \ldots, v_k) \ni v_{k+1} = X^{k+1}v \\
U_k &\subseteq U_{k+1} \\
X^{k+2}v &= X(X^{k+2}v) \in X \cdot \text{span}(V_0, v_1, \ldots, v_k) \\
&= \text{span}(Xv_0, Xv_1, \ldots, Xv_k) \\
&= \text{span}(v_1, v_2, \ldots v_{k+1}).
\end{align*}
\]
So indeed, \( U_{k+1} = U_{k+2} = U_{k+3} = \cdots \), a similar proof.
\[
\begin{align*}
U_0 &= \text{span}(v_0) \\
U_1 &= \text{span}(V_0, v_1) \Rightarrow \{v_0, v_1\} \text{ linearly independent.} \\
U_2 &= \text{span}(V_0, v_1, v_2) \Rightarrow \{v_0, v_1, v_2\} \text{ linearly independent.} \\
&\vdots \\
U_k &= \text{span}(V_0, v_1, v_2, \ldots v_k) \Rightarrow \{v_0, v_1, v_2, \ldots v_k\} \text{ linearly independent.}
\end{align*}
\]
So, \( \{v_0, v_1, \ldots, v_k\} \) is a basis for \( U_k \), so that \( \dim U_k = k + 1 \). (Note: \( U_0 \neq U_1 \neq U_2 \neq \cdots \neq U_k \))

(3) We will show by induction on \( i = 0, 1, \ldots, k \) that
(a) \( U_i \) is \( Y \)-invariant, i.e. \( Y(U_i) \subseteq U_i \).
(b) the matrix of $Y|_{U_i}$ relative to the basis $\{v_0, v_1, \ldots, v_i\}$ is upper triangular with
$\lambda(Y)$ on the diagonal. In particular $Yv_i = \lambda(Y)v_i + U_{i-1}$

$$\begin{pmatrix}
\lambda(Y) & \cdots & * \\
\vdots & \ddots & \vdots \\
0 & \cdots & \lambda(Y)
\end{pmatrix}$$

So, now proving this using induction.

(I) $i = 0$ then $U_0 = \mathbb{C}v \subset W$. $Yv = \lambda(Y)v$ so $U_0$ is $Y$-invariant. $Yv_0 \in \lambda(Y)v_0$

(II) The induction step. Assume that parts (a) and (b) are true for indices $0, 1, \ldots, i-1$ instead of $i$. Need to prove parts (a) and (b) for $i$ itself. $Yv_j \in \lambda(Y)v_j + U_{j-1}$

(**) for $j = 0, 1, \ldots, i-1$ - by our assumption.

$$YV_i = YX^i v = YXX^{i-1}v = YXv_{i-1} = XYv_{i-1} + [Y, X]|_{v_{i-1}}$$

$$\in X(\lambda(Y)v_{i-1} + U_{i-2}) + \lambda([Y, X])v_{i-1} + U_{i-2} = \ldots$$

But inductive assumption is made for all $Y \in \mathfrak{h}$, in particular for $[Y, X]$ instead of $Y$.

$$\cdots \subseteq \lambda(Y)v_i + U_{i-1} + \lambda([Y, X])v_{i-1} + U_{i-2}$$

so $Yv_i \in \lambda(Y)v_i + U_{i-1}$, i.e. (**) is true for $j = 0, \ldots, i$.

(**) is true for $v_0, \ldots, v_i$ - basis in $U_i$

(a) $Y(v_j) \subseteq \text{span}(v_j, v_{j-1}, \ldots, v_0)$ true.

(b) also true.

(4) trace($Y|_{U_i}$) = $(i + 1)\lambda(Y)$ true for all $Y \in \mathfrak{h}$ also

(***)

$$\text{trace} \left( \left[ Y, X \right] \bigg|_{U_i} \right) = (i + 1)\lambda([Y, X])$$

put $i = k$. $X$ preserves $U_k$ - by construction and $Y$ preserves $U_k$ - induction.

$$[Y, X]|_{v_k} = (YX - XY)|_{v_k} = Y|_{U_k}X|_{U_k} - X|_{U_k}Y|_{U_k}$$

$$\Rightarrow \text{trace} \left( [Y, X]|_{v_k} \right) = 0 = (k + 1)\lambda([Y, X])$$

$$\Rightarrow \lambda([Y, X]) = 0$$

□

by (***)
6. Representations of Lie Algebras

Definition: For any Lie algebra $\mathfrak{g}$ a representation of $\mathfrak{g}$, over $\mathbb{C}$, is any Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ where $V$ is any vector space over $\mathbb{C}$.

Remark. if $\dim V = n < \infty$ then $\mathfrak{gl}(V) = \mathfrak{gl}_n(\mathbb{C})$.

Remark. it is possible that $\ker \varphi = \{X \in \mathfrak{g} | \varphi(X) = 0\} \neq \{0\}$. For instance, one can define a representation by setting $\varphi(X) = 0$ for all $X \in \mathfrak{g}$. (called trivial representation).

Definition: $\varphi$ is faithful if $\ker \varphi = \{0\}$.

Theorem 9 (Ado). If $\dim \mathfrak{g} < \infty$ then there exists a faithful representation of $\mathfrak{g}$ (without proof).

Definition: suppose $\varphi_1 : \mathfrak{g} \to \mathfrak{gl}(V_1)$ and $\varphi_2 : \mathfrak{g} \to \mathfrak{gl}(V_2)$ are representations of $\mathfrak{g}$. Then $\varphi_1 \oplus \varphi_2$ is a representation of $\mathfrak{g}$ defined as follows:

$$V = V_1 \oplus V_2$$
$$\varphi : \mathfrak{g} \to \mathfrak{gl}(V_1 \oplus V_2)$$
$$\varphi(X)(u_1, u_2) = (\varphi_1(X)u_1, \varphi_2(X)u_2)$$

Comment: suppose $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation and $U \subseteq V$ such that $\varphi(X)U \subseteq U$ or $\varphi(X)u \in U$ (for $u \in U$). Then $X \in \mathfrak{g}$ $\varphi(X)|_U \in \mathfrak{gl}(U)$; $X \mapsto \varphi(X)|_U$ is a homomorphism $\mathfrak{g} \to \mathfrak{gl}(U)$ another representation.

Definition: $\varphi$ is an irreducible representation if the only subspaces $U \subseteq V$ such that $\varphi(X)U \subseteq U$ for all $X \in \mathfrak{g}$ are the trivial ones, $U = V$, $\{0\}$.

For example, question 17 gives a full list of all irreducible representations of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.

7. Revision

7.1. Main definitions in order of appearance

- Lie algebra $\mathfrak{g}$ (3 axioms)
- Abelian Lie algebra ($[X, Y] \equiv 0$)
- Homomorphism of Lie algebras ($\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$)
- Isomorphism (homomorphism & bijection)
- Direct sum of Lie algebras ($\mathfrak{g}_1 \oplus \mathfrak{g}_2$)
- Derivations of Lie algebras ($\delta : \mathfrak{g} \to \mathfrak{g}$ linear, Leibniz rule) - inner (ad $\mathfrak{z} : \mathfrak{g} \to \mathfrak{g}$), outer derivations.
- A central element $Z$ of $\mathfrak{g}$, the centre $\mathcal{Z}(\mathfrak{g}) = \{Z \in \mathfrak{g} | Z \text{ central}\}$.
- Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ then $[X, Y] \in \mathfrak{h}$ for all $X, Y \in \mathfrak{h}$
• Ideal of $g \supseteq h$ subspace such that $[X, Y] \in h$ for all $X \in g$ and $Y \in h$
• Canonical homomorphism $\pi : g \to g/\mathfrak{h}$ onto.
• Derived Series of $g$: $g \supseteq g' \supseteq g'' \supseteq \ldots$
• Solvable Lie algebra (derived series terminates).
• Radical $\mathcal{R}(g)$ of any Lie algebra (maximal solvable ideal)

**Theorem 10.** $g/\mathcal{R}(g)$ - semisimple

• Semisimple Lie algebra has no solvable ideals except for $\{0\}$.

**Theorem 11.** $g$ semisimple $\Rightarrow g = g_1 \oplus \cdots \oplus g_k$ where $g_1 \ldots g_k$ are simple.

• Simple Lie algebra $g$ (no ideals, except for $\{0\}$, $g$ itself).
• Classification of simple Lie algebras ($A$, $B$, ... in terms of Dynkin diagrams).

**Example 7.1.** $\mathfrak{gl}_4 \mathbb{C} \supset g = \{ A = \begin{pmatrix} X & Z \\ Y & 0 \end{pmatrix} : X, Y \in \mathfrak{sl}_2 \mathbb{C}, Z \text{ any } 2 \times 2 \text{ matrix} \}, \ [A, B] = AB - BA$

$$\mathcal{R}(g) = \left\{ \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} : Z \text{ any } 2 \times 2 \text{ matrix} \right\} = \{ A | X = Y = 0 \}$$

$$g/\mathcal{R}(g) = \left\{ \begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix} + \begin{pmatrix} 0 & \tilde{Z} \\ 0 & 0 \end{pmatrix} : \tilde{Z} \text{ any } 2 \times 2 \text{ matrix} \right\}$$

$$= \left\{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} + \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} : Z \text{ any } 2 \times 2 \text{ matrix} \right\}$$

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} \tilde{X} & 0 \\ 0 & \tilde{Y} \end{pmatrix} - \begin{pmatrix} \tilde{X} & 0 \\ 0 & \tilde{Y} \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} = \begin{pmatrix} X\tilde{X} - \tilde{X}X & 0 \\ 0 & Y\tilde{Y} - \tilde{Y}Y \end{pmatrix}$$

another good representation.