Hurwitz - Brill - Noether Theory

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$C = \text{curve}$

Q: Describe all maps $C \rightarrow \mathbb{P}^r$ of degree $d$.

Given a line bundle $L$ on $C$, let $s_0, s_1, \ldots, s_r \in H^0(C, L)$ basis.

$C \rightarrow \mathbb{P}^r$

$p \mapsto [s_0(p), s_1(p), \ldots, s_r(p)]$

Given a map $\varphi: C \rightarrow \mathbb{P}^r$,

$\varphi^* \mathcal{O}_{\mathbb{P}^r}(1)$ is a line bundle on $C$.

Brill - Noether varieties:

$W^r_d(C) := \{ \mathcal{L} \in \text{Pic}^d(C) | \dim H^0(C, \mathcal{L}) \geq r \}$
Thm: Let $C$ be a general curve of genus $g$. Then $W^r_d(C)$ is:
1) equidimensional of dim $g-(r+1)(g-d+r)$ [Griffiths-Harris '80]
2) smooth away from $W^{r+1}_d(C)$ [Gieseker '82]
3) irreducible if dim > 0 [Fulton-Lazarsfeld '81]

What if $C$ is not general?

$C$ is equipped with a map $\pi: C \rightarrow \mathbb{P}^1$ of degree $k$.

In other words, $(C,\pi)$ is an element of the Hurwitz space $\mathcal{H}_{g,k}$.

Ex: $g=5$ $k=3$ $W^1_4(C)$

\[ (-2,-2,1) \quad \mathcal{E}g_3 + p_1 \quad p \in C^3 \]

\[ (-3,-1,1) \]

\[ (-3,0,0) \quad \mathcal{E}K - g_3 - q \quad q \in C^3 \]
Let $L$ be a line bundle on $C$.  
\[ \pi^* L = \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^1}(m_i) = \mathcal{O}(\mathbf{u}) \]
\[ \mathbf{u} = (m_1, m_2, \ldots, m_k) \].

$\mathbf{u}$ is the splitting type of $L$.

\[ \mathbf{W}^u(C, \pi) := \left\{ \xi \in \text{Pic}(C) \mid \pi^* L \cong \mathcal{O}(\mathbf{u}) \right\} \]

**Thm:** Let $(C, \pi) \in H_{g,k}$ be general.

Then $\mathbf{W}^u(C, \pi)$ is ...

1. equidimensional of dim \( g - \sum_{i=1}^{k} \max(2, m_i - m_j - 1) \)
   
   [H. Larson ’19, Cooil-Powell ’20]

2. smooth [H. Larson ’19]

3. irreducible when dim $\geq 20$ [LLCV ’20]

**Degeneration Arguments**

$H_{g,k}$ is irreducible, so any nonempty open subset is dense.

All conditions above are open.

It's enough to find one $(C, \pi)$ satisfying $\mathcal{C}$).

Instead of finding such a pair, you degenerate to a singular curve.
Prop: Let \((C, \pi)\) be this chain of \(g\) elliptic curves with torsion profile \(k\). Then

\[
W^\mu(C, \pi) = \bigcup_{t \text{ uniform displacement tableau on } \lambda(\mu)} T(t)
\]

\[
\lambda(\mu) := \{ (x, y) \in \mathbb{N}^2 \mid \exists m \in \mathbb{Z} \text{ s.t.} \ x \leq h^0(\mathbb{P}^1, \Theta(\mu + m)) \\
\text{and} \ y \leq h^1(\mathbb{P}^1, \Theta(\mu_3 + m)) \} \tag{5}
\]

Ex: \(g = 5, k = 3\) \[
\mu_3 = (-3, 1, 1)
\]

Def: A \(k\)-uniform displacement tableau \(t\) on a \(A\)-labeled \((a)\).
partition $\lambda$ with alphabet $E_g$

is a function $t: \lambda \rightarrow [g]$ satisfying:

1) $t$ is increasing across rows and down cols.

2) if $t(x,y) = t(x',y')$, then $\gamma - x \equiv y' - x' \pmod{k}$.

Given a k-udf $t$ on $\lambda(E_g^\infty)$, one can construct a family of limit linear series $T(t)$.

- if $i$ appears in the tableau, its position determines the line bundle on component $i$.

- if $i$ does not appear in the tableau, then the $1^\text{st}$ of the $i^\text{th}$ component can be anything.

$\dim T(t) = \# \text{ of symbols in } [g] \text{ that do not appear in } t.$

$\lambda(E_g^\infty)$ is an example of a $k$-core.

$k$-cores $\leftrightarrow$ infinite Coxeter systems $A_k$.

Strong and weak order on the set of $k$-core.

- Standard $\leftrightarrow$ saturated chain in the Yang lattice.

- Yang tableaux $\leftrightarrow$ $k$-uniform.

- Displacement $\leftrightarrow$ the weak order lattice.
Tableaux of k-cores

- equidimensionality \( \leftrightarrow \) weak order lattice
- codimension \( \leftrightarrow \) rank of \( X(n) \) in this lattice
- connectedness \( \leftrightarrow \) "braid moves"
- containment \( \leftrightarrow \) strong order on k-cores