A smooth compactification of $M_{2,n}(P^n,d)$ via Gorenstein singularities & log geometry

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0. Motivations

The spaces 

$$\overline{M}_{g,n}(\mathbb{P}^n,d) = \left\{ F: (c_1,\ldots,c_n) \to \mathbb{P}^n \mid \deg F = d, \ell_g(c_1) = g, \ |\text{Aut}(F)| \leq \infty \right\}$$

are fundamental in enumerative geometry.

For example, for any $Y \subseteq \mathbb{P}^n$ smooth

$$\overline{M}_{g,n}(Y,d) \subseteq \overline{M}_{g,n}(\mathbb{P}^n,d)$$

and we would like to compute $G_{n,g}(Y)$ applying intersection theoretic methods on $\overline{M}_{g,n}(\mathbb{P}^n,d)$

For $g=0$, $\overline{M}_{0,n}(\mathbb{P}^n,d)$ is smooth and indeed we can computing $G_{0,n}(Y)$ integrating suitable Chern classes on $[\overline{M}_{0,n}(\mathbb{P}^n,d)]$
For \( g \geq 1 \), \( \overline{M}_{g,n}(\mathbb{P}^r,d) \) has many irreducible components of different dimensions intersecting.

\[ g=1 \quad r=2 \quad d=3 \]

\[ \text{dim} 3 \quad \text{dim} 10 \quad \text{dim} 3 \]

\[ 0 \rightarrow 9 \quad 0 \rightarrow 5 \]

\[ \text{boundary \ germs} \quad \text{closure \ of \ smooth \ locus} \quad \text{main \ component} \]

\[ \Rightarrow \text{difficult to compute } G(X) \text{ and } G(Y) \]

- the invariants \( G(X) \) also count lower genus curves due to boundary contributions

Motivating question:

can we find a smooth compactification

\[ \overline{M}_{g,n}(\mathbb{P}^r,d) \subset \sqrt{Z}_{g,n}(\mathbb{P}^r,d) \]?

1. Previous work in this direction

For \( r=1 \) Volkil - Zinger, Hu - Li 2003/2010
Describe an explicit sequence of blow-ups (based on the knowledge of local equations) which resolve
\[
\overline{M}_{1}^{\text{main}}(\mathbb{P}_1^2,d) \quad \text{and} \quad \overline{M}_{2}^{\text{main}}(\mathbb{P}_1^2,d)
\]
\[
\uparrow \text{bin.} \quad \uparrow \text{bin.}
\]
\[
\overline{M}_{1}^{\text{main}}(\mathbb{P}_1^2,d) \quad \overline{M}_{2}^{\text{main}}(\mathbb{P}_1^2,d)
\]
smooth

In 2017 Rongpan Chen, Jonsson, Păun, and Wise present a different way to resolve
\[
\overline{M}_{1}^{\text{main}}(\mathbb{P}_1^2,d)
\]
using Smyth genus 1 Gorenstein singularities & Log geometry

2. genus 2 Gorenstein singularities
By a Gorenstein curve we mean a curve with a canonical line bundle.

Genus 2 isolated singularities (Bottistella)

Type I | Type II
\[ m = 1 \quad < \quad y^2 - x^3 \]

\[ m = 2 \quad \mathbf{K} \quad x (y^2 - x^3) \]

\[ m = 3 \quad \mathbf{K} \quad \begin{cases} A^3 & \text{if} \quad z = 2 (x - y), \\ z^3 - x y & \end{cases} \]

\[ m \geq 4 \quad \mathbf{K} \quad \begin{cases} x \in A^m \quad < x (y - x^2), \\ x (x - y), \\ x \in A^m - 1 \quad < x (x - y), \\ x (x - y), \\ x (x - y) \end{cases} \]

\[ * = \text{special branches} \quad 0 = \text{twin branches} \]

\textit{Genus 2 non isolated} \textit{Gorenstein singularities} (just one example)

\[ C = \begin{cases} T_1 \\ \vdots \\ R \end{cases} \quad \text{local equation around this point} \quad \frac{\partial c y^4}{\partial x y} \]

\textit{R} is a \textit{non reduced curve} \textit{with} \textit{R} \textit{red} \textit{\&} \textit{P} \textit{1}

and \[ (\mathcal{L}_{P_1 \cap \mathcal{R}})^2 = 0 \text{ in } \mathcal{R} \]

we call \[ \mathcal{R} \text{ e ribbon} \]

and \[ \overline{C} \text{ e teiled Ribbon} \]

if \[ \mathcal{O}_R \text{ fits in} \quad \mathcal{O}(m^3) \rightarrow \mathcal{O}_R \rightarrow \mathcal{O}_{P_1} \rightarrow 0 \]

\[ \rightarrow \quad \text{R}^2 = 2 \]
If I have a 1, 0 or high enough degree on these for genus 2 curves

\[ \Rightarrow \quad \dim H^1(\mathcal{E}, \mathcal{L}) = 0 \]

**Cohomological Lemma**

(a) \( \mathcal{E} \) genus 2 Gorenstein isolated singularity

without any rational tail nodally attached

\[ \Rightarrow \quad \text{a line bundle } \mathcal{L} \text{ on } \mathcal{E} \text{ with:} \]

\[ \deg \mathcal{L} \mid_{c_i} \geq 0 \quad \forall c_i \in \mathcal{E} \]

\[ \deg \mathcal{L} \mid_{c_0} > 0 \quad \text{for (at least one) special } c_0 \]

\[ \text{tot} \deg \mathcal{L} \geq 3 \]

\[ \text{has } H^1(\mathcal{E}, \mathcal{L}) = 0 \]

(b) \( \mathcal{E} \) is a twisted ribbon and \( \mathcal{L} \) is line bundle s.t.

\[ \deg \mathcal{L} > 0 \quad \text{on at least two tails} \]

\[ \quad \text{on } \mathcal{R}_{\text{red}} \text{ and at least one tail} \]

\[ \Rightarrow \quad H^1(\mathcal{E}, \mathcal{L}) = 0 \]

3. The main idea behind our approach

Recall/Notice that \( \text{obstructions} \)

For \( F \in \overline{M}_{2,n}(\mathbb{P}^r, 1) \) arise when

\[ H^1(\mathcal{E}, F^* \mathcal{O}(1)) \neq 0 \]
This happens when:

1. $F$ contracts a genus $1$ or $2$ subcurve $Z \subset C$

2. when $F^{\overline{\phi}}$ is the minimal genus $2$ has degree $2$

Neive idea

- In case (1), replace $\overline{Z}$ is obtained collapsing $Z$ to a point

- In case (2) replace in $\overline{C}$ we replaced $\overline{\phi}$ with a Ribbon

$\Rightarrow$ We expect $\overline{F}^* \mathcal{O}(1)$ to satisfy the

$H^4(\overline{Z}, \overline{F}^* \mathcal{O}(1)) = 0$

$\Rightarrow$ NO OBSTRUCTIONS
Issues with the naive idea

0. The map \( C \to \Sigma \) has moduli: what singularity \( \Sigma \) should have is not determined by \( (C, F) \)

But depend from the choice of smoothing family

\[
\begin{array}{ccc}
\text{model curve} & & \text{family} \\
\circ & \overset{a \to f \to p} \rightarrow & \Sigma
\end{array}
\]

\[
\begin{array}{c}
\circ \overset{b} \rightarrow \Sigma
\end{array}
\]

If I have a smoothing of \( C \) \( \Delta \) is a semiample $\Delta$

\[
\underline{\Sigma} = \text{Proj} \left( \oplus \pi_1^* L \right)
\]

\( \Sigma \) is Gorenstein \( \Rightarrow \) I can fill $\Delta$ contracted locus

\[
\left[ R = \text{Weil}(D) \left( \Sigma, \Delta \right) \right]
\]

away from what is contracted

1. The factorization can only exist for smoothable maps

2. To have a moduli functor we need
Goal becomes

\[ \text{construct a moduli space } \tilde{X} \xrightarrow{e} M_{2,n} \]

\[ \text{sufficient information to construct } \tilde{E} \]

\[ \xrightarrow{\text{partial destabiliz.}} e^* \tilde{E} \rightarrow \tilde{E} \]

\[ \xrightarrow{\text{Einstein}} \tilde{E} \]

\[ \xrightarrow{\text{wt(\tilde{E}) satisfying the hp of the coh. lemme}} \]

4. Log Tropical curves and admissible covers

**DEF:** A log scheme \((X, M_X)\) is \(\tilde{X}\) scheme

\[ M_X \text{ is a sheaf of monoids} \]

\[ M_X \xrightarrow{\alpha} \mathcal{O}_X \]

s.t.

\[ \alpha^*(\mathcal{O}_X) = \mathcal{O}_X \]

\[ \text{ghost sheaf} \]
Example 0
\[ X = \text{Spec } \mathbb{k}, \text{ plus your favourite monoid } \mathbb{P} = \mathbb{N} \oplus \mathbb{N} \]
\[ M_X = \mathbb{k}^* \oplus \mathbb{N}^n \rightarrow \mathbb{k} \]
\[ (\lambda, \sigma) \rightarrow \begin{cases} \lambda & \text{if } \sigma = 0 \\ \sigma & \text{otherwise} \end{cases} \]

Example 1
\[ X \text{ smooth projective variety } D \subseteq X \text{ normal crossing divisor} \]
\[ M_X = \{ f \in \mathcal{O}_X | F|_{X \setminus D} \in \mathcal{O}_X^* \} \]
\[ M_{X,x} = \mathbb{N} \]
\[ k = \# \text{pts of } D \text{ meeting } x \]

Curves

Let \( \overline{M}_{g,1,n} \) there is a canonical log structure induced by the boundary divisor \( (\overline{M}_{g,1,n}, \overline{M}_D) \)
Family of log smooth curves

\[ (t, M_t) \]

work as follows:

\[ (S, M_S) \]

\[ C = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array} \]

\[ \text{more or less} \]

\[ \text{shadow of} \quad xy = e^n \]

\[ e^x, e^y \]

\[ \log x, \log y \]

\[ \| S' = N \|

Remark: If we consider the canonical structure on the moduli space of curves

\[ \Rightarrow \text{ on one point} \quad s \rightarrow M_{g,n} \]

\[ \overline{M}_S = \# \text{nodes of } C_S \]

The ghost part of the log structure is recorded in the tropicalization

A tropical curve is \( \text{Trop}(C, M_C) \)

\[ E = \text{dual graph}(C) \]

\[ S' \]

\[ \text{ghost sheaf of} \quad \sqrt{\text{the base}} \]
We think of $S, S'$ as lengths of these edges.

If $\overline{M}_s = N$ you can think that $S, S'$ are remembering the speed of $e$ smoothing of $C$.

Notice that in this case $S$ and $S'$ are comparable.

$E$ is a leveled graph.

Admissible covers

A smooth genus $2$ curve is hyperelliptic:

$$c = \begin{array}{c} \circ \end{array} \overset{2:1}{\rightarrow} \mathbb{P}^1$$

- $\gamma$ is branched in $6$ pts (called Weierstrass).
- We say $\gamma$ and $\bar{\gamma}$ are conjugated if $\gamma(p) = \bar{\gamma}(p)$.
- $\gamma^* \mathcal{O}(1) = \mathcal{O}$.
Instead of compactifying $\mathcal{M}_{2,n}$ to $\mathcal{M}_{3,n}$ we can consider the

$\mathcal{M}_{2,n} \subset \mathcal{A}_{2,n} \subset \text{admissible covers}$

$\uparrow$

$\psi: C \rightarrow \mathcal{T}$ c model, $g=2$, $T$ model $g=0$

- $\psi$ is 2:1 branched on 6 smooth $\mathcal{T}$

and

- Around nodes $\mathcal{C}([x, y, e]) \leftarrow \mathcal{C}([x, y, e])$

$\frac{xy-e^i}{s-t-e}$

is $s \rightarrow y$.

- $T$ marked with $P_1, \ldots, P_n + branch$

pts $b_1, \ldots, b_6$ is all-stable

Examples

Also on $\mathcal{A}_{2,n}$ there is a natural log-structure
\[ \mathcal{M}_{2,n} = \mathcal{M}_{2,n} \oplus \mathcal{M}_{0,n+6} / \{S, 0\} \sim \{0, \delta'\} \]

Tropical version

- edge length of corresponding edges are the same
- edge length of their image in \( \mathcal{T} \)
- \( S^T = 2S \)

As for the curve, we can consider variations of \( \mathcal{A}_{2,n} \)

\[ \sim \quad \mathcal{A}_{2,n} \quad \Leftarrow \quad \text{vertices are decorated with weight and weighted st. condition} \]

Piecewise linear function on the tropicalization

On log schemes, we have a distinguished class of line bundles coming from \( \mathcal{T}(X, \overline{M}_X^{\delta_0}) \)

Recall \( 0 \to \mathcal{O}_X^* \to \mathcal{M}_X^{\delta_0} \to \overline{M}_X^{\delta_0} \to 0 \)
\[ \rightarrow \Gamma (\mathcal{M}^{\mathcal{G}_0}_X) \rightarrow H^4 (\mathcal{O}_X^*) \]

If \( X \rightarrow S \) is log smooth curve

\[ \Gamma (\mathcal{M}^{\mathcal{G}_0}_X) = \left\{ \text{Piecewise linear functions} \right\} \]

\[ \lambda : \mathcal{E} \rightarrow \mathcal{M}_S \]

\[ x(v_1) - \lambda(v_2) \in \mathbb{Z} \]

\[ \mathbb{Z} \]

nodes associated to \( e_1, e_2, e_3 \)

\[ \Rightarrow \mathcal{O}_c (\lambda) \mid_{c_v} = \mathcal{O} (2p_1 + p_2 - p_3) \]

slopes of \( \lambda \)

5. Hein Results & Idea of the proof

**Theorem 1 (Bottistelle, -)**

There exists a log stable model (birational)

\[ A_{2,1}^{u_t} \rightarrow A_{2,1}^{u_t} \rightarrow M_{2,1}^{u_t} \]

which is smooth. Moreover
Theorem 2 (Bottstelle, -)

Let $\mathcal{V}^2_\mathbb{Z}(X/Y)$ the moduli space parametrizing

- A family $F \rightarrow Y$ of $\mathbb{Z}$-algebraic log admissible covers

- A map $f: \mathcal{X} \rightarrow X$ satisfying the factorization
\( \Rightarrow (1) \forall \mathbb{Z}_{\geq n}(X, \partial) \) is a proper DM-stack.

(2) For \( X = \mathbb{P}^n \), this is unobstructed.

**Proof of (1)**

Let \( \mathcal{E} \) be \( n \)-plane in \( \mathcal{E} \), \( \mathcal{E} \) smoothing family, and \( \mathcal{E}^{ss} \rightarrow \mathcal{E} \) generic.

\[ \Rightarrow \phi^* \omega_{\mathcal{E}^{ss}} = \omega_{\mathcal{E}}(\lambda) \] where \( \lambda \) is defined on \( \mathcal{E} \) and looks like this.

\[ e = 2 \] in type II and \( p = 0 \).
\[ S = 2^N \]

\[ e = 3 \] in type I

and the toil is attached to e W

\[ \text{section } d(C_{sp, co}) / d(\text{other branches}) \]

\[ \sqrt{3} \]

\[ \sqrt{2} \]

\[ \frac{1}{2} \]

\[ \text{if we started from a Ribbon} \]

To give \( e \) starting from \( e \rightarrow T \)

\[ \text{choose a special vertex } v_0 \text{ which tell us what the slopes of } \lambda \text{ are} \]

\[ \text{compare } \lambda(w), \lambda(v) \text{ for each vertex} \]

\[ \text{information included in } \Delta z/w \]

\[ e_2 \]

\[ e_2 = e_2 \text{ splitting} \]

\[ u = e_2 \text{ interaction} \]
As the lengths vary we need to move the special vertex.

Subdividing \( \mathcal{M}_{2,n} \) we move \( \nu \) aligned on each cone we can compare the value at the vertices and know that the special branch is the unique one with slope \( \neq 1 \).

Set \( \mathcal{E} \) and \( \mathcal{F} \).