Genus two reduced quasi-map invariants for CY$^3$ complete intersections (Joint with J. Oh, M.-L. Li)

Today's plan — Sanghyeon Lee, KIAS —

1. Moduli space of stable quasi-maps (= quasi-map space) (focused on genus 2, target is $P^n$) + $p$-fields

2. Local charts, local equations for genus 2 quasi-map spaces with $p$-fields and its desingularization

3. Normal cone computation and splitting of virtual cycle one of them gives "reduced" invariants

4. Computations to obtain standard versus reduced formula.

5. Further projects
Moduli space of stable quasi-maps

- Genus $g$, degree $d$ stable quasi-map to $X \subseteq \mathbb{P}^n$ is defined by:
  - $C$: genus $g$ nodal curve with marked points $P_1, \ldots, P_k \in \mathbb{P}^n$
  - $L$: degree $d$ line bundle on $C$
  - $u = (u_0, \ldots, u_k) \in H^0(C, L)^{\oplus n+1}$, base locus is finite, does not meet special points (nodes and marked points)

- The image of $C \xrightarrow{u} \text{Tot}(L^{\oplus n+1})$ lies in
  $L \otimes C C(X) \subseteq \text{Tot}(L^{\oplus n+1})$

- $W_C (\Sigma P_i) \otimes L^\epsilon$ is ample for any $\epsilon > 0$
  - any genus $0$ component of $C$
  - need at least $3$ special points to be stable
  - If $k = 0$ (no marked points), no rational tail is allowed

There exist a moduli space $\overline{Q}_{g,k}(X, d)$ parametrizing those quasi-maps, proper DM stack.

- It carries relative P.O.T $E \to L \overline{Q}/B$

$C \xrightarrow{\pi} \overline{Q}_{g,k}(X, d)$: canonical curve, $L$ : univ. line bundle
\[ U = (U_0, \ldots, U_n) : \text{univ. section of } L \]
\[ \Rightarrow U \text{ induce a map } \mathcal{E} \to L \otimes E C(X) \]
\[ \Rightarrow \mathcal{E} := (R^i \pi_*(\text{Hom}(LU, OC)[1]))^\vee \]
If \( X = \mathbb{P}^n \), \( \mathcal{E} = ((R^i \pi_* L)^\oplus n + 1)^\vee \)

From this P.D.T, we construct virtual cycle
\[ \left[ \overline{Q}_{g,k}(X, d) \right]^{vir} \in A_{\text{dim}} \left( \overline{Q}_{g,k}(X, d) \right) \]
\[ \text{vdim} = (\text{dim } X - 3)(-g) + k - d \text{ [line]} \cdot W_X \]

Quasi-map invariant \( \left< \psi_{i_1}^a, S_{i_1}, \ldots, \psi_{i_n}^a, S_{i_n} \right>^X \) is defined by
\[ \int [\overline{Q}_{g,k}(X, d)]^{vir} \psi_{i_1}^a \psi_{i_2}^a \psi_{i_3}^a \cdots \psi_{i_n}^a \in \mathbb{Q} \]

\(- \text{p-fields}\)

Next we consider stable quasi-maps with \( p \)-fields
It is given by \((C, L, u, p = (p_1, \ldots, p_m))\)
\((C, L, u)\) is a stable quasi-map to \( \mathbb{P}^n \)
\[ p_i \in H^0(C, W_C \otimes L_{l_i}) \quad (l_1, \ldots, l_m \in \mathbb{Z}_{>0}, \text{ fixed}) \]

There is a moduli space \( \overline{Q}^p_{g,k}(\mathbb{P}^n, d) \to DM \text{ stack} \)
parametrizing stable maps with \( p \)-fields
It carries relative P.O.T $E \to \mathbb{L}$. For
\[
\overline{Q}^p \pi \to B, \quad \mathbb{L} \otimes E := (R^\pi_1 \mathcal{L})^\otimes \oplus \bigoplus_{i=1}^m R^\pi_1 \mathcal{L}^i
\]

Assume that $X \subseteq \mathcal{L}$, i.e., defined by $X = \{ f_1 = \ldots = f_n = 0 \}$
\[\text{deg} f_i = \lambda_i\]

- There is a cosection $h^i (E, \mathcal{L}) \to \mathcal{O}$
  induced from $p$-fields and eqns $f_1, \ldots, f_m$

- Degeneracy locus of $G$ (where $G$ vanishes) is
\[
\overline{Q}_{g, k} (X, d) \subseteq \overline{Q}_{g, k} (\mathcal{L}) \subseteq \overline{Q}^p_{g, k} (\mathcal{L})
\]
\[
\Rightarrow \left[ \overline{Q}_{g, k} (X, d) \right]^{\text{vir}} = (-1)^d \left( \sum \lambda_i \right) + m(-q) \left[ \overline{Q}^p_{g, k} (\mathcal{L}) \right]^{\text{vir}}
\]
\[\text{Kim-Oh, Chang-Li}\]

$\Rightarrow$ we can use $\overline{Q}^p_{g, k} (\mathcal{L})$ instead of $\overline{Q}_{g, k} (X, d)$

for computing $\left< \psi_1^{x_1}, \delta_1, \ldots, \psi_{x_k}^{x_k}, \delta_k \right>_g$

- Some closed locus in $\overline{Q}(\mathcal{L}) = \overline{Q}_{2,0} (\mathcal{L})$ (or $\overline{Q}^p (\mathcal{L})$)
we describe some closed loci in $\overline{Q}(\mathcal{L})$ (or $\overline{Q}^p (\mathcal{L})$)
  will be used later.
1. Let $Q^0 = \overline{Q}(P^m)$ be an open subset of pairs $(C, D)$ such that $C$ is smooth.

2. Let $\overline{Q}^0$ be the closure of $Q^0$, called main component.

3. $Z_1 = \{ \text{\textbf{diagram of shaded components}} \}$

4. $Z_2 = \{ \text{\textbf{diagram of shaded components}} \}$

5. $Z_3 = \{ \text{\textbf{diagram of shaded components}} \}$

Note that $Z_2 \subset Z_1$, $Z_2 \cap Z_3 = \emptyset$.

$Z_1 \cap Z_3 = \{ \text{\textbf{diagram of shaded components}} \}$

$Z_1, Z_2, Z_3$ are all smooth substack of $\overline{Q}(P^m)$. 


2. Local charts and local equations for quasi-map space (with p-fields) and its desingularization.

From now on we consider the case $g=2$, $k=0$, $d \geq 3$

$X \subseteq \mathbb{P}^n$ $CY$ $3$ complete intersection

defined by $X = \{ f_i = \ldots = f_m = 0 \}, \quad \text{deg} f_i = n_i = n-3$

Consider an Artin stack $m_{\text{div}} \Rightarrow m_{\text{div}}$

parametrizing pairs $\left( C, D \right)$

$\text{genus} \ g$ nodal curve $\text{deg} \ d$ effective div on $C$

with $k$ marked points

$(s_0, \ldots, s_n)$

Consider $x = \left[ (C, L, s, \rho) \right] \in \overline{Q}^p(\mathbb{P}^n) = \overline{Q}^p_{2,0}(\mathbb{P}^n, d)$

Assume $x \in \mathbb{Z}_1 \cup \mathbb{Z}_2 \cup \mathbb{Z}_3$ "boundary"

Since since base locus of $s$ is finite, we may assume $s_0 \neq 0$. Let $D = \mathbb{Z}(s_0)$

Construct affine local charts of $\overline{Q}^p(\mathbb{P}^n, d)$

around $x$.

First, we choose a small affine nbd $V \xrightarrow{\text{sm}} m_{\text{div}}$

around $(C, D)$. Let $D_V$ be a canonical divisor in the univ. curve $\mathbb{C}_V$ of $V$. 
Now consider a section space

\[ \mathcal{F}(D_v) = \Pi_k \mathcal{O}(D_v) \bigoplus_{s=(s_1, \ldots, s_n)} \left( \bigoplus_{i=1}^m \mathcal{O}_{e_v n}(-l_i D_v) \right) \]

Let \( U \subset \text{open} \mathcal{F}(D_v) \) s.t. for \( y \in V \), \( (s_1, \ldots, s_n) \in \mathcal{F}_y | y \in V \),

\[ (Cy, \mathcal{O}(D_y), s=(s_1, \ldots, s_n), p=(p_1, \ldots, p_m)) \]

is a stable quasi-map.

It induces a smooth morphism \( U \longrightarrow \overline{Q}(P^n) \),
which is a smooth around \( X \).

So we compute \( \Pi_k \mathcal{O}(D_v) \)

( for \( p \)-fields, \( \Pi_k (\mathcal{O}_{e_v n}(-l_i D_v) \), we can compute similarly using serve duality).

Since base points are finite, we may assume \( S_0 \neq 0 \).

Let \( D = 2(S_0) = a_1 p_1 + \cdots + a_mp_m \) (\( a_i \in \mathbb{Z}_{>0} \), \( \sum a_i = d \))

Since \( x \) is contained in the boundary, we observe \( \kappa^0(C, \mathcal{O}(2p_i)) \geq 2 \)

we can find a meromorphic \( f \) on \( C \), which has double pole at \( p_i \) (If \( f \) has a simple pole, it induces a deg 1 map \( \mathbb{C} \longrightarrow \mathbb{P}^1 \times \)).
\[ g \text{ induces a } 2:1 \text{ morphism } \mathbb{C} \xrightarrow{g} \mathbb{P}^1 \]

For unramified point \( t \in \mathbb{P}^1 \), \( f^{-1}(t) = p_i' + p_i'' \), \( 2p_i \sim p_i' + p_i'' \). We may assume that \( p_i', p_i'' \) distinct from other \( p_j \neq p_i \).

So that in finite steps, we can construct an isom
\[
\mathcal{O}(\mathbb{C}, \mathcal{O}(p_i') + \mathcal{O}(p_i'')) \cong \mathcal{O}(\mathbb{C}, \mathcal{O}(p_i'))
\]

\[ \Rightarrow \exists \ s_0' \in H^0(\mathbb{C}, \mathcal{O}(p_i') + \mathcal{O}(p_i'')) \text{ correspond to } \]
\[ s_0 \in H^0(\mathbb{C}, \mathcal{O}(\mathcal{O}(p_i'))) \]

\[ \Rightarrow \exists s_0' = \mathcal{O}(\mathbb{C}, \mathcal{O}(p_i') + \mathcal{O}(p_i'')) \]

If \( U \) is small enough, we can extend this to define an isom \( u : V \xrightarrow{\sim} W \)

\[ \text{s.t. for } t = (C_t, D_t) \in V, \ u(t) = (C_u(t), D_u(t)), \]
\[ C_u(t) = C_t, \ D_u(t) = p_i' + \ldots + p_i'^d \]

\[ \Rightarrow \text{Can. div } \mathcal{D}_w \subset \mathcal{C}_w \text{ is of the form } \]
\[ \mathcal{D}_w = D_1 + \ldots + D_d, \ D_i \cap D_j = \emptyset \text{ for } i, j \]
We have \( \mathcal{F}(D_V) \subseteq \mathcal{U}^* \mathcal{F}(D_W) \).

So, for simple notation, we let \( W = V \).

Use methods in Hu-Li-Niu’s paper

\[
\begin{array}{c}
\text{for local chart & equations for} \\
g \geq 2 \text{ stable map & its desing}
\end{array}
\]

We choose additional sections \( x_1, x_2, \mathcal{B} : V \to \mathcal{C} \),

don’t intersect to each others.

If \((C, D) \in \mathcal{E}V\) is of the form

\[ \begin{pmatrix} a_1 & a_2 \end{pmatrix} \]

then img of \( x_1, x_2 \), \( a_1, a_2 \) lies in each genus \( 1 \) comps, separated

Then we have

\[
R^* \pi_* \mathcal{O}(D_1 + \cdots + D_d) = R^* \pi_* \mathcal{O}(D_1 + \cdots + D_d - \mathcal{B}) \oplus \mathcal{O}_V,
\]

\[
R^* \pi_* \mathcal{O}(D_1 + \cdots + D_d - \mathcal{B}) \cong \left[ \begin{array}{c}
\pi_* \mathcal{O}(D_1 + \cdots + D_d + A_1 + A_2 - \mathcal{B}) \\
\rightarrow \mathcal{O}_V \oplus \mathcal{O}_V \\
\mathcal{O}_V \oplus \mathcal{O}_V
\end{array} \right]
\]

from some cohomology computations.

Also we have decomp

\[
\pi_* \mathcal{O}(D_1 + \cdots + D_d + A_1 + A_2 - \mathcal{B}) \cong \bigoplus_{i=1}^d \mathcal{O}_V(D_i + A_1 + A_2 - \mathcal{B})
\]
So that we can express \((\text{ev}_{A_1} \oplus \text{ev}_{A_2})\) as a \(2 \times d\) matrix

\[
\begin{pmatrix}
C_{11} & C_{12} & \ldots & C_{1d} \\
C_{21} & C_{22} & \ldots & C_{2d}
\end{pmatrix},
\]

Note that each column express

\[
\begin{pmatrix}
C_{1i} \\
C_{2i}
\end{pmatrix},
\]

\[
\Pi_{\mu} \left( U \left( D_{i} + A_{1} + A_{2} - B \right) \right) \text{ev}_{A_1} \oplus \text{ev}_{A_2} \mapsto \text{ev}_{U} \oplus \text{ev}_{V}
\]

\[
\Pi_{\mu}
\]

Ex)

\[
C = \begin{pmatrix}
a_1 & p & a_2 \\
C_1 & R & C_2
\end{pmatrix} \Rightarrow \text{ev}_{A_1} = 0
\]

\[
\Rightarrow \text{ev}_{A_1} = \mathbb{C} \in \mathbb{C}^k
\]

If the node \(z_1\) is smoothed

\[
\Rightarrow \begin{pmatrix}
a_2 \\
C_1 & p & C_2
\end{pmatrix} \Rightarrow \text{ev}_{A_1} = \mathbb{C} \in \mathbb{C}^k
\]

\[
\Rightarrow \text{ev}_{A_1} = \mathbb{C} \in \mathbb{C}^k
\]

\[
\Rightarrow \text{let } \tau_1 \text{ be the node-smoothing parameter on } V
\]

\[
\tau_1 = \{ \tau_1 = 0 \} \text{ is the locus where node } z_1 \text{ is not smoothed}
\]

\[
\Rightarrow Z_i(\text{ev}_{A_i}) = \{ \tau_i = 0 \}, \text{ ev}_{A_1} = c \cdot \tau_1 \quad (c \in \mathbb{P}(U^*), r \in Z > 0)
\]
we can check $r=1$! (see Hu-Li, pages 1 desing for detail)

Similarly, we can check $\psi(d) = c' \cdot r_2$ ($c' \in \Gamma(\mathcal{O}_V^*)$)

Since $d \geq 3$, we may choose $p_1, p_2$ are not conjugate

the $2 \times d$ matrix becomes

\[
\begin{pmatrix}
T_1 & 0 & 0 \\
0 & T_2 & 0
\end{pmatrix}
\]

via some basis change of $\mathcal{O}_V^d$ and $\mathcal{O}_V^{\otimes 2}$.

Similarly, if $x = (C, D) \in V$ is of the form

\[
\begin{matrix}
\ast \\
\ast
\end{matrix}
\]

\[\Rightarrow \]

$2 \times d$ matrix is of the form (via suitable basis change)

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & T_1 & 0 \\
0 & 0 & T_2
\end{pmatrix}
\]

If $x = (C, D) \in V$ is of the form

\[
\begin{matrix}
\ast \\
\ast
\end{matrix}
\]

we can check $2 \times d$ matrix is of the form:

\[
\begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & a_1 T_1 + b_1 T_2 & a_2 T_1 + b_2 T_2 & \ldots & a_{d-1} T_1 + b_{d-1} T_2
\end{pmatrix}
\]

\{ $a_i T_1 + b_i T_2 = 0$ \} is a locus where $p_1, p_2$ conj
We blow-up this locus \( \{ z_1 = z_2 = 0 \} \subset V \)
in the level of \( M^w \)
\[ \mathcal{V} := V \times_{M^w} M^w, \quad \mathcal{U} := U \times_{M^w} M^w. \]

Let \( E \) be a local parameter of \( \mathcal{V} \) corresponding to
the exc divisor \( E \subset \mathcal{V} \)

\( \Rightarrow \) pull-back of \( \text{ev}_{\mathcal{V}} \circ \) ev\( \mathcal{V}_2 \), \( \tilde{\text{ev}}_{\mathcal{V}} \circ \tilde{\text{ev}}_{\mathcal{V}_2} \)
is of the form:
\[
\begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]

Similarly, \( \text{Rmap} \mathcal{V}/\mathcal{V} \times \mathcal{O}(-\text{li} \; \mathcal{V}) \)
is:
\[
\begin{bmatrix}
\mathcal{O}_{\mathcal{V}}^2 & \mathcal{O}_{\mathcal{V}} \oplus \text{li} \; \mathcal{V}
\end{bmatrix}
\]
given by transpose of \( 2 \times \text{li} \; \mathcal{V} \) matrices with same form as above.

Using these results, we can compute
\( \text{Tot}(T_{\mathcal{V}} \mathcal{O}(\mathcal{V})) \) as subspace of \( \text{Tot}(\mathcal{O}_{\mathcal{V}} \oplus \mathcal{O}_{\mathcal{V}}^2) \)
\( \cong \mathcal{V} \times \mathcal{O}^{2m + 2m} \)
and find equations \( \mathcal{O}_{\mathcal{V}} \oplus \mathcal{O}_{\mathcal{V}}^2 \) is an oblique bundle for \( \mathcal{V}/\mathcal{V} \).
Let \( \left( X_1, X_2, \ldots, X_{1d} \right) \) be coordinates of \( \mathbb{C}^d \)
\( \left( X_{21}, X_{22}, \ldots, X_{2d} \right) \)
\( \vdots \)
\( \left( X_{n1}, X_{n2}, \ldots, X_{nd} \right) \) be coordinates of \( \mathbb{C}^n \)
\( \left( X_{m1}, X_{m2} \right) \) be coordinates of \( \mathbb{C}^m \)

\[ \Rightarrow \text{rewrite } X_{1i} \sim X_{m1} \text{ by } X_i \sim X_{m1}, \]
\[ X_{12} \sim X_{m2} \text{ by } Y_i \sim Y_{m2}. \]

If \( V \) is a sub around \( x = (C, D) \) of type

\[ \text{Tot} (I) \subset \mathbb{C}^{nd+2m} \]

is defined by equations:

\[ X_1 T_1 = \cdots = X_{m1} T_1 = 0, \quad Y_1 T_2 = \cdots = Y_{m1} T_2 = 0 \]

Then the normal cone \( \mathbb{C}^{nd+2m} \)
\[ \text{Tot} (I) / \mathbb{C}^{nd+2m} \]

can be computed by the following:

Let \( \text{Spec} (R) = \mathbb{C}^{2d+2m} \)

\[ \Rightarrow \text{ideal } : (X_1 T_1, \ldots, X_{m1} T_1, Y_1 T_2, \ldots, Y_{m1} T_2) \subset R \left[ x_1, \ldots, x_m, \right. \]
\[ \left. y_1, \ldots, y_{m2} \right] \]

Cone parameter: \( a_1, \ldots, a_{m1}, b_1, \ldots, b_m \)
3. Normal cone computation and splitting of virtual cycle

\[ \text{Normal cone : } R[C_{x_1, \ldots, x_{\text{utm}}, y_1, \ldots, y_{\text{utm}}}[d_1, \ldots, d_{\text{utm}}, \beta_1, \ldots, \beta_{\text{utm}}] \]

\[ \overset{\wedge}{R} \rightarrow \left( \frac{y_i \beta_j - y_j \beta_i}{y_i \varepsilon_2 - x_i \varepsilon_1} \right) \]

Normal cone has 3 irreducible components:

1. Over \( \{ x_1 = \ldots = x_{\text{utm}} = y_1 = \ldots = y_{\text{utm}} = 0 \} \subset \text{Tot}(\mathcal{F}(\mathcal{D}v)) \)

\[ \overset{\wedge}{R} \otimes_{\overset{\wedge}{R}} \bar{R}/(x_1, \ldots, x_{\text{utm}}, y_1, \ldots, y_{\text{utm}}) \]

\[ \cong \left( \frac{\bar{R}/(x_1, \ldots, x_{\text{utm}}, y_1, \ldots, y_{\text{utm}})}{[d_1, \ldots, d_{\text{utm}}, \beta_1, \ldots, \beta_{\text{utm}}]} \right) \]

Let \( \text{Spec } \bar{R}/(x_1, \ldots, x_{\text{utm}}, y_1, \ldots, y_{\text{utm}}) \cap \tilde{\mathcal{U}} := \tilde{\mathcal{U}}_{\text{red}} \)

Over \( \tilde{\mathcal{U}}_{\text{red}} \), \( \mathcal{N}_{\text{red}} \times \mathbb{C}^{\text{dim} + 2m} |_{\tilde{\mathcal{U}}_{\text{red}}} \cong \tilde{\mathcal{U}}_{\text{red}} \times \mathbb{C}^{2(\text{utm})} \)

2. Over \( \{ x_1 = \ldots = x_{\text{utm}} = 0, \varepsilon_2 = 0 \} \)

\[ \text{and } \{ y_1 = \ldots = y_{\text{utm}} = 0, \varepsilon_1 = 0 \} \)

\[ \overset{\wedge}{R} \otimes_{\overset{\wedge}{R}} \bar{R}/(x_1, \ldots, x_{\text{utm}}, \varepsilon_2) \]

\[ \cong \frac{\bar{R}/(x_1, \ldots, x_{\text{utm}})}{[y_1, \ldots, y_{\text{utm}}][d_1, \ldots, d_{\text{utm}}][\beta_1, \ldots, \beta_{\text{utm}}]} \]

\[ \left( y_i \beta_j - y_j \beta_i \right) \]

\[ \Rightarrow \text{fiber over } \text{Spec } \bar{R}/(x_1, \ldots, x_{\text{utm}}) \] are affine cone of the blow-up \( \text{Bl}_{_{\text{red}}} / \mathcal{A}_{\text{utm}} \Rightarrow \text{irreducible} \)
Let \((\text{Spec } \mathbb{R}/(X_1, \ldots, X_{\text{dim}}), Y_1, \ldots, Y_{\text{dim}})) \cap U = U_1 = C_{U/V \times \text{Spec } \mathbb{R}} \text{ glued to an}
\text{inred. comp of the cone over } \mathbb{R}^1 \mathbb{P}(\mathbb{P}^n)
\text{U}_1 \text{ glued to an inred comp of } \mathbb{P}_1 \mathbb{P}(\mathbb{P}^n) \subset \mathbb{P}_1 \mathbb{P}(\mathbb{P}^n)\text{ red}
\text{corresponds to } \begin{array}{c}
\text{\includegraphics[width=2cm]{cone.png}}
\end{array}
\text{\text{reduced genus quasi-map}}
\text{defined in Mu-Lin Li's 2019 paper similarly defined as } \mathbb{Q}^1 \mathbb{P}_1 \mathbb{P}(\mathbb{P}^n)
\text{over } \{T_1 = T_2 = 0 \}
\text{similarly defined as } \mathbb{Q}^1 \mathbb{P}_1 \mathbb{P}(\mathbb{P}^n)\text{ red}
\text{fiber over } \text{Spec } \mathbb{R}/(T_1, T_2)
\text{is a product of}
\text{affine cones of the blow-up } \mathbb{B}_0 A^{\text{dim}}
\Rightarrow \text{inredible}

\text{Let } U_2 := \text{Spec } \mathbb{R}/(T_1, T_2)[X_1, \ldots, X_{\text{dim}}, Y_1, \ldots, Y_{\text{dim}}] \cap U
\Rightarrow C_{U/V \times \text{Spec } \mathbb{R}} \text{ glued to an inred comp. over }
\text{\includegraphics[width=2cm]{cone.png}}
\text{corresponds to } \begin{array}{c}
\text{\includegraphics[width=2cm]{cone.png}}
\end{array}
\text{gluing of } U_2
Similarly, we can do similar cone computations for other local charts of $M_{\text{div}}$. 

By considering some base changes via $\mathcal{B} \to M_{\text{div}}$

we obtain local computation of relative intrinsic normal cone $\mathcal{C}(\mathbb{P}^n)/\mathcal{B}$

$\Rightarrow \mathcal{C}(\mathbb{P}^n)/\mathcal{B} = \mathcal{C}^{\text{red}} \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$

$\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ lies over boundaries $\bar{Z}_1, \bar{Z}_2, \bar{Z}_3 \subset \mathbb{P}^n$

Take localized Gysin map, we have

$$b_! \mathcal{O}_V^{1/2}/\mathbb{Q}^p(\mathbb{P}^n), \mathcal{C}(\mathbb{P}^n)/\mathcal{B} \rightarrow \mathbb{Q}^p(\mathbb{P}^n)$$

from Costello's push-forward formula

$$= (-1)^{d(\pi_1\pi_2)} \mathcal{C}(\mathbb{P}^n)^{\text{vir}}$$

$$= [\mathcal{Q}(\mathbb{P}^n)_{\text{red}}]^{\text{vir}} + [\mathcal{Q}(\mathbb{P}^n)_1]^{\text{vir}} + [\mathcal{Q}(\mathbb{P}^n)_2]^{\text{vir}}$$

gives reduced quasi-map inv $\Rightarrow$ boundary cycles
Gives splitting of virtual cycle.

All cycles supported in \( \overline{Q}(X) \)

We can check

\[
[\overline{Q}^p([p^n])_{\text{red}}]_{\text{vir}} \cong b \times \mathbb{O}_N[1] \left[ \overline{Q}([p^n])_{\text{red}} \right]^{\text{smooth}}
\]

\[
N = m \sum \mathbb{L}_k \left| \overline{Q}([p^n])_{\text{red}} \right| \rightarrow \text{loc. free from } R^{\text{top}}\mathbb{L} \text{ computation}
\]

Cosection \( \sigma : \text{Ob}(\mathbb{E}^* \mathbb{V}) \rightarrow \mathbb{O} \)

induces cosection \( \sigma : N \rightarrow \mathbb{O} \)

\[ \Rightarrow \text{quantum Lefschetz property for genus 2 reduced quasi-map inv.} \]

4. Computations to obtain standard versus reduced formula

\[ \langle \chi_{2,d} \rangle - \langle \chi_{2,d}^{\text{red}} \rangle \text{ comes from degree of boundary cycles.} \]

Briefly explain how to compute \( [\overline{Q}^p([p^n])_{\text{red}}]_{\text{vir}} \)

\[ \text{supp. in } \{ \bullet \bullet \bullet \} \]
Recall the local computation above,
\( \mathcal{C}_2 \) locally given by

\[
\text{Spec} \left( \frac{\mathbb{R}(\mathcal{C}_2, \mathcal{C}_2)}{[X_i, \ldots, X_{\text{utm}}, \underbrace{Y_i, \ldots, Y_{\text{utm}}}] / (X_k d_j - X_j d_i, \ Y_i b_j - Y_j b_i)} \right)
\]

over \( \mathbb{R}(\mathcal{C}_2, \mathcal{C}_2) \) \( [X_i, \ldots, X_{\text{utm}}, \underbrace{Y_i, \ldots, Y_{\text{utm}}}] \)

\( \cong \mathbb{Q}^{(\text{utm})^2} \)

It can be considered by the following consider

\[
\Psi : \text{Spec} \left( \mathbb{R}(\mathcal{C}_2, \mathcal{C}_2) \right) \times \mathbb{C}^2 \xrightarrow{\Psi_1, \Psi_2} \text{Spec} \left( \mathbb{R}(\mathcal{C}_2, \mathcal{C}_2) \right) \times \mathbb{C}^{(\text{utm})} \times \mathbb{C}^{(\text{utm})}
\]

\[ \Rightarrow \text{im} \Psi = \mathcal{C} \]

Let \( \text{Spec} (\mathcal{R}) := \mathcal{B} \mathcal{B}_2 (\mathcal{B} \mathcal{B}, \text{Spec} (\mathcal{R})) \),

\[
\mathcal{B}_1 = \{ X_i = \ldots = X_{\text{utm}} = 0 \}
\]

\[
\mathcal{B}_2 = \{ Y_i = \ldots = Y_{\text{utm}} = 0 \}
\]

Let \( E_1, E_2 \) be corresponding exc divisors

\( \Psi \) induces an injection \( \text{Spec} (\mathcal{R}) \times \mathbb{C}^2 \xrightarrow{\Phi} \text{Spec} (\mathcal{R}) \times \mathbb{C}^2 \) (twisting \( E_1, E_2 \))
For the projection \( P: \hat{\mathcal{R}} \to \mathcal{R} \), we have

\[ p_x(\text{im} \hat{\psi}) = C \]

We observe \( \psi \) glued to the natural morphism

\[ P^* N_{\mathcal{R}^2 / \mathcal{M}^w} \xrightarrow{\psi} \left( \mathcal{C} \right)^{\mathcal{P}(\mathcal{P})/\mathcal{M}^w} \xrightarrow{\text{obs. sheaf}} \]

\[ \left( \mathcal{P} : \mathcal{C}^{\mathcal{P}(\mathcal{P})/\mathcal{M}^w} \right) = \mathcal{H} \left( \mathcal{E} \mathcal{C}^{\mathcal{P}(\mathcal{P})/\mathcal{M}^w} \right) |_{\mathcal{C}^{\mathcal{P}(\mathcal{P})/\mathcal{M}^w}} \]

Because it is locally induced from the ideal inclusion

\[ (T_1, x_1, \ldots, T_3, x_{3+m}, T_2, t_1, \ldots, T_3, x_{3+m}) \subset (T_1, T_3) \]

\[ \Rightarrow \mathcal{C}^{\mathcal{P}(\mathcal{P})/\mathcal{M}^w} \xrightarrow{P^*} \mathcal{R}^2 / \mathcal{M}^w \]

For \( P^*: \mathcal{C}^{\mathcal{P}(\mathcal{P})/\mathcal{M}^w} \Rightarrow \mathcal{R}^2 / \mathcal{M}^w \)

and exc. divisor \( E \),

\[ \Rightarrow \psi \text{ induces an injection} \]

\[ \hat{\psi}: p^* P^* N_{\mathcal{R}^2 / \mathcal{M}^w}(E) \to p^* \text{obs. sheaf} \]

\[ \text{Set, } p_x \text{ Im } \hat{\psi} = \mathcal{C}^{\mathcal{P}(\mathcal{P})/\mathcal{M}^w} \xrightarrow{\text{obs. sheaf}} \]

\[ \mathcal{C}^{\mathcal{P}(\mathcal{P})/\mathcal{M}^w} \]
Then

\[
0^! \left. C^* \Omega^{\bar{a}_{obs}}(\mathbb{P}^{n}/\mathbb{P}^{m}/\mathbb{P}^{w}(E)) \right|_{\bar{a}_{obs}^{\hat{a}}(\mathbb{P}^{n})^2} = P_\mathcal{X} \left( C_{\text{top}} \left( \frac{C(p^* \Omega^{\bar{a}_{obs}}/p^* \mathcal{U}^* N_m^2/\mathcal{M}^w(E))}{C(p^* \mathcal{U}^* N_m^2/\mathcal{M}^w)} \right) \right.
\]

\[
= P_\mathcal{X} \left( \frac{C(p^* \Omega^{\bar{a}_{obs}})}{C(p^* \mathcal{U}^* N_m^2/\mathcal{M}^w)} \right) \text{rk obs}_2 - 2
\]

expressed by universal bundles at nodes

By some further cohomology comp (similar to Zinger’s)
we obtain closed formula for \( \langle \rangle_{2,d} - \langle \rangle_{2,d} \)

For exact argument, we need “localized” Gysin way,
it use some modified method (see L.-Oh 20’)

We can compute other boundary cycles similarly.
As a result, we obtain the closed formula:

\[
\langle \rangle_{2,d} - \langle \rangle_{2,d}
\]

\[
= \frac{-1}{192} \left( \int [\bar{Q}_{0,2}(X,d)] \text{vir} \Psi_1 \Psi_2 + 2(1-EV_{2}^* C_{2}(TX)) \right)
\]

\[
- \frac{1}{24} \int [\bar{Q}_{1,1}(X,d)_{\text{red}}] \Psi_1
\]
5. Further projects

- Using this formula + Wall crossing btw quasi-map inv and GW-inv

We are going to give another way for computation of genus 2 GW-inv of quintic 3-folds and another CY3 C.i C|P^m

- We believe that all these methods directly applied to the case X= CY3 C.i C|P^m x ... x |P^mnr.
Also going to compute genus 2 GW-inv in this case

- We assumed d≥3 to avoid "conjugacy point" issue. If D = P1+P2 then dimension of sections H^*(C, D) differs by whether P1, P2 is conjugate or not

⇒ "conjugacy parameter" on mod inv affects to (2x d)-matrix we considered above

But, we want to remove this assumption for full-generality and for genus 2 GW computation in all degree. So we will also try to remove d≥3 assumption

-End-