Consider the map $T : \mathcal{P}_2(\mathbb{F}) \rightarrow \mathbb{F}^2$ given by $T(p) = (p(0), p(1))$.

1. (4 points) Show that $T$ is a linear map.

   $\forall p, q \in \mathcal{P}_2(\mathbb{F}), \quad T(p + q) = (p + q)(0), (p + q)(1) = (p(0) + q(0), p(1) + q(1)) = T(p) + T(q)$

2. (4 points) Find the kernel (i.e. the null space) $\ker T$ of $T$.

   $p(t) = a_0 + a_1 t + a_2 t^2 \in \ker T \\
\iff \begin{cases} a_0 = 0 \\ a_0 - a_1 + a_2 = 0 \end{cases}$

   Thus, $\ker T = \text{Span} \{ t + t^2 \}$

3. (2 points) Find $\dim \ker T$. Is $T$ injective? Why?

   By 2), $\dim \ker T = 1 \neq 0 \implies T$ is not injective.

4. (2 points) Find rank $T$ (i.e. $\dim \text{range} T$). Is $T$ surjective? Why?

   By Rank-Nullity Theorem and Part 3),
   $\text{rank } T = \dim \mathcal{P}_2(\mathbb{F}) - \dim \ker T = 3 - 1 = 2 = \dim \mathbb{F}^2$

   $\Rightarrow T$ is surjective.

5. (3 points) Find the matrix of $T$ with respect to the standard bases $1, t, t^2$ of $\mathcal{P}_2(\mathbb{F})$ and $(1, 0), (0, 1)$ of $\mathbb{F}^2$.

   $T(1) = (1, 1) \quad \Rightarrow \quad M(T) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$
   $T(t) = (0, -1)$
   $T(t^2) = (0, 1)$