Research Statement

Jingchen Niu

I am interested in the moduli problems and the resolution of singularities in symplectic topology and algebraic geometry. In particular, my research involves the study of the resolution of singularities related to the moduli of stable maps and the study of the real Gromov-Witten invariants.

Moduli problems are of central importance in symplectic topology and algebraic geometry. Among them, the moduli $M_g$ of stable genus $g$ pseudo-holomorphic maps to a target space $X$ play a prominent role. They are of the main objects of interest in the theory of complex GW-invariants, which arise from Gromov’s work [Gr] on pseudo-holomorphic curves and Witten’s work [Wi] on $\sigma$-models in physics and are often related to integer counts of curves in $X$. Moduli spaces, however, are often singular; some of them, including the moduli of stable maps, possess arbitrary singularities [V]. The resolution of singularities is arguably one of the hardest problems in symplectic topology and algebraic geometry [Hi1, Hi2, deJ].

Suppose the target space $X$ is $\mathbb{P}^n$. The stable map moduli $M_g$ is smooth when $g = 0$ and singular when $g \geq 1$ and $n \geq 2$. In the genus 1 case, Vakil and Zinger in [VZ] construct a desingularization $\tilde{M}_1$ of the primary component $\overline{M}_1$ of $M_1$, which leads to an effective computation of the GW-invariants of complete intersections and ultimately to the proof [Z2] of the prediction of [BCOV] for genus 1 GW-invariants of the quintic 3-fold. In [HL], Hu and Li provide an algebro-geometric approach to the desingularization $\tilde{M}_1$ of $M_1$. Both constructions involve a sequence of smooth blowups. In the genus 2 case, Hu, Li, and I further develop the blowup method to finally establish a resolution $\tilde{M}_2$ of $M_2$ in [HLN]. For arbitrary genus, the blowup construction of a possible resolution of $M_g$ may seem formidable. To obtain a more abstract and geometric approach, Hu and I construct the moduli of curves with twisted fields in the genus 1 and 2 cases [HN1, HN2], which in turn provide modular interpretations of the resolutions $\tilde{M}_1$ and $\tilde{M}_2$. The new method does not require any blowup procedure and appears promising in the arbitrary genus case. These works are described in Section 1 below.

The theory of real GW-invariants has long lagged behind the complex theory. These invariants, first defined in genus 0 cases in [We1, We2], should count pseudo-holomorphic maps from symmetric Riemann surfaces commuting with the involutions on the domain and the target. In [GZ], the real GW-invariants are defined in all genera for many symplectic manifolds, including all odd-dimensional complex projective spaces and the quintic 3-fold. In [NZ], Zinger and I establish a formula that transforms real GW-invariants of many symplectic 3-folds into signed integer counts of smooth real curves. Our formula gives rise to many enumerative results, e.g. in $\mathbb{P}^3$ there are at least 10 genus 2 degree 7 real curves passing through 7 general pairs of conjugate points and at least 40 genus 5 degree 8 real curves passing through 8 general pairs of conjugate points. As a byproduct of our formula, we obtain implications for certain Hodge integrals. This work is described in Section 2 below.
1 Moduli of stable maps and Modular Resolutions

For $g \geq 1$, let $\overline{M}_g(P^n, d)$ be the moduli space of degree $d$ genus $g$ stable maps to $P^n$. Although the moduli $\overline{M}_g(P^n, d)$ determines a rational homology class (virtual fundamental class), it can be arbitrarily singular for general $g$ and $d$. Moreover, with $\pi: C \to \overline{M}_g(P^n, d)$, $f: C \to P^n$ denoting the universal family of $\overline{M}_g(P^n, d)$, the direct image sheaf
$$\pi_* f^* O_{P^n}(k) \to \overline{M}_g(P^n, d)$$
is not locally free. This raises the following question.

Question 1 (Q1). Does there exist a resolution $\mathcal{E}$ of $\overline{M}_g(P^n, d)$ such that

\begin{itemize}
  \item the irreducible components of $\mathcal{E}$ are smooth and intersect transversally, and
  \item for each irreducible component $N$ of $\mathcal{E}$ with $r_N \neq 0$ the pull-back universal family, the direct image sheaf $\pi_N^* f^* O_{P^n}(k)$ is locally free?
\end{itemize}

An affirmative answer to Q1 would not only lead to a resolution of singularities in a moduli satisfying the “Murphy’s law”, but also allow direct application of Atiyah-Bott localization formula to the computation of GW-invariants of complete intersections. For $g = 1$, the affirmative answer has been confirmed by Vakil and Zinger via symplectic topology [VZ] and by Hu and Li via algebraic geometry [HL].

Theorem 2 ([HLN]). The answer to Question 1 is affirmative for $g = 2$.

Our approach in [HLN] improves the technique of [HL]. We study the degeneracy loci of certain direct image complexes and construct blowups based on such loci to locally diagonalize the direct image complexes. Compared to its genus 1 counterpart, the blowup procedure in [HLN] is considerably longer, in part because of the various topological types of genus 2 curves and the existence of Weierstrass and conjugate points.

In order obtain a more abstract and geometric approach towards Q1 for arbitrary $g$, Hu and I introduce the notion of the moduli of curves with twisted fields. In the genus 1 case, let $M_1^{wt}$ be the algebraic moduli stack consisting of the pairs $(C, w)$ of genus 1 nodal curves $C$ with non-negative weights $w \in H^2(C, \mathbb{Z})$, meaning $\sum w = 0$ for all irreducible $\Sigma \subset C$. There exists a natural morphism
$$\overline{M}_1(P^n, d) \to M_1^{wt}, \quad [C, w] \mapsto [C, c_1(u^* O_{P^n}(1))].$$

In [HN1], Hu and I construct a smooth algebraic moduli stack $M_1^{tf}$ of weighted genus 1 nodal curves with twisted fields, along with a forgetful morphism $M_1^{tf} \to M_1^{wt}$.

Theorem 3 ([HN1]). The fiber product
$$\overline{M}_1^{tf}(P^n, d) = \overline{M}_1(P^n, d) \times_{M_1^{wt}} M_1^{tf}$$
is a moduli stack and is isomorphic to the resolution $\mathcal{E}$ of $\overline{M}_1(P^n, d)$ obtained via blowups in [VZ, HL].
Theorem 3 provides a new modular interpretation of the blowup space \( \overline{M}_1(P^n, d) \), in addition to the interpretation in [RaSW] via logarithmic geometry. In our approach, we first stratify the stack \( \mathcal{M}_1^{wt} \) by the dual graphs of the curves, then add a fiber product of certain projective bundles over each stratum of \( \mathcal{M}_1^{wt} \), and finally patch them together to form \( \overline{M}_1^{tf} \): the fibers over a given point \( x \in \mathcal{M}_1^{wt} \) are the \( twisted \ fields \) over \( x \). Such construction can actually be generalized to a much wider range of moduli stacks. For example, in the genus 2 case, let \( \mathcal{P}_2 \) be the relative Picard stack over the stack of genus 2 nodal curves. There exists a natural morphism

\[
\overline{M}_2(P^n, d) \to \mathcal{P}_2, \quad [C, u] \mapsto [C, u^* \mathcal{O}_{P^n}(1)].
\]

In [HN2], Hu and I construct a smooth algebraic moduli stack \( \mathcal{P}_2^{tf} \) of genus 2 nodal curves with line bundles and twisted fields, along with a forgetful morphism \( \mathcal{P}_2^{tf} \to \mathcal{P}_2 \).

**Theorem 4 ([HN2]).** The fiber product

\[
\overline{M}_2^{tf}(P^n, d) = \overline{M}_2(P^n, d) \times_{\mathcal{P}_2} \mathcal{P}_2^{tf}
\]

is a moduli stack. Moreover, there exists a natural resolution \( \overline{M}_2^{tf}(P^n, d) \to \overline{M}_2(P^n, d) \) that provides an affirmative answer to Question 1.

The resolution \( \overline{M}_2^{tf}(P^n, d) \to \overline{M}_2(P^n, d) \) in Theorem 4 factors through the blowup resolution \( \overline{M}_2(P^n, d) \to \overline{M}_2(P^n, d) \) in [HLN].

**Further research.**

One direction of my further research is to extend the modular constructions in [HN1, HN2] to the arbitrary genus case, i.e. to construct \( \overline{M}_g^{tf}(P^n, d) \) for arbitrary \( g \) and show they provide affirmative answers to Question 1. This would lead to very useful results in the theory of resolution of singularities, as well as a possible way to compute the GW-invariants of complete intersections for \( g \geq 2 \). For the latter, a good start point would be to find the localization data for \( \overline{M}_2^{tf}(P^n, d) \) or \( \overline{M}_2(P^n, d) \), compute the genus 2 GW-invariants for quintic 3-folds, and compare the answers with the results in [GmJR].

Another direction of further research is to apply the method in [HN1, HN2] to desingularize other moduli spaces, such as the moduli of stable quotients (with marked points), or the moduli of stable maps to targets other than \( P^n \).

I would also like to work on the topology of the moduli of curves with twisted fields, such as their cohomology, so as to know the topology of the resolutions in Question 1 even just for lower genus cases. So far there has not been any known result, but I am very interested in such studies.

I am also interested in a related study of the moduli of stable maps for arbitrary compact symplectic manifold \( X \). Let \( B \in H_2(X; \mathbb{Z}) \), \( J \) be a generic almost complex structure, and

\[
\overline{M}_{g,k}(X, B; J) \supset \overline{M}_{g,k}^0(X, B; J)
\]

be the moduli space of stable genus \( g \) pseudo-holomorphic maps with \( k \) marked points in the homology class \( B \) and the subspace of maps from smooth domain curves, respectively. In general \( \overline{M}_{g,k}(X, B; J) \) can be highly singular and does not contain \( \overline{M}_{g,k}^0(X, B; J) \) as a dense subset (for any choice of \( J \)) unless \( g = 0 \). This raises the question of whether there exists a natural subspace

\[
\overline{M}_{g,k}^0(X, B; J) \subset \overline{M}_{g,k}(X, B; J)
\]
containing $\mathcal{M}^0_{g,k}(X, B; J)$ such that $\mathcal{M}^0_{g,k}(X, B; J)$ is dense in $\mathcal{M}_{g,k}(X, B; J)$ (for sufficiently regular $J$). For $g = 1$, such a closed subspace is constructed in [Z1]. For $g = 2$, I am working on constructing $\mathcal{M}_{2,k}^0(X, B; J)$ in [N], which should give rise to the reduced genus 2 GW-invariants arising from $\mathcal{M}_{2,k}^0(X, B; J, \nu)$ with certain perturbations $\nu$ of the Cauchy-Riemann equation. It would be very interesting to study their relation with the usual GW-invariants.

2 Real GW-invariants

A compact real symplectic manifold $(X, \omega, \phi)$ consists of a compact symplectic manifold $(X, \omega)$ and an anti-symplectic involution $\phi$ (i.e. $\phi^* \omega = -\omega$). The almost complex structures $J$ are chosen so that $\phi^* J = -J$. A symmetric surface $(\Sigma, \sigma)$ is a (possibly nodal) Riemann surface with an orientation-reversing involution $\sigma$. A real pseudo-holomorphic map is a pseudo-holomorphic map $u : \Sigma \to X$ such that $u \circ \sigma = \phi \circ u$; its image in $X$ is a real curve. The main object in the real GW-theory of [GZ] is the moduli space $\mathcal{M}_{g,l}(X, B; J)^\phi$ of stable real pseudo-holomorphic maps from genus $g$ symmetric surfaces with $l$ conjugate pairs of marked points in the homology class $B \in H_2(X; \mathbb{Z})$. The real moduli space is potentially non-orientable, but for a large family of real symplectic manifolds, including $\mathbb{P}^{2n-1}$ and the quintic 3-fold, a real orientation of [GZ] on $X$ endows $\mathcal{M}_{g,l}(X, B; J)^\phi$ with an orientation and a virtual class and thus gives rise to real GW-invariants of arbitrary genus for $X$.

In the complex GW-theory, the Fano case of the Gopakumar-Vafa prediction (conjectured in [P], proved in [Z3]) implies that the positive-genus GW-invariants for a Fano class $B$ ($c_1(B) > 0$) of a compact symplectic 3-fold can be canonically expressed in terms of integer counts of curves. In [NZ], Zinger and I establish a similar formula that transforms real positive-genus GW-invariants for a Fano class $B$ of a compact real-oriented symplectic 3-fold into signed integer counts of smooth real curves. These integer invariants provide lower bounds for the actual counts of such curves.

Suppose $X$ is a compact real-oriented symplectic 3-fold and $B \in H_2(X; \mathbb{Z})$ is a Fano class. For $g, l \in \mathbb{Z}_{>0}$, let $\mathcal{M}_{g,l}^s(X, B; J)^\phi \subset \mathcal{M}_{g,l}(X, B; J)^\phi$ be the subspace consisting of simple (i.e. not multiple-covered) real maps from smooth domains. The constraints $\mu_i \in H^*(X; \mathbb{Z})$, $1 \leq i \leq l$, are taken so that their dimensions sum up to $c_1(B) + 2l$. A tuple $f = (f_i : Y_i \to X)_{i=1}^t$ of pseudocycle representatives for the Poincare duals of $\mu_1, \ldots, \mu_t$ cuts out a space

$$\mathcal{M}_{g,l}^s(X, B; J)^\phi = \{ ([u, (z_i^+, z_i^-)]_{i=1}^l, (y_i)_{i=1}^l) \in \mathcal{M}_{g,l}^s(X, B; J)^\phi \times \prod_{i=1}^l \mathcal{Y}_i : u(z_i^+) = f_i(y_i) \ \forall \ i \}.$$

In [NZ], we show that for a generic choice of $J$ (depending on $g$ and $B$) and each $h \leq g$

- the moduli space $\mathcal{M}_{h,l}^s(X, B; J)^\phi$ consists of regular maps and
- for a generic choice of $f$, $\mathcal{M}_{h,l}^s(X, B; J)^\phi$ is a finite set of regular pairs $([u, (z_i^+, z_i^-)]_{i=1}^l, (y_i)_{i=1}^l)$ such that $u$ is an embedding.

The signed cardinality $\pm |\mathcal{M}_{h,l}^s(X, B; J)^\phi|$ is proved to be independent of the choice of $J$ and $f$ and can be denoted by $E_{h,B}(\mu_1, \ldots, \mu_t)$. This is a signed integer count of genus $h$ real curves.
Theorem 5 ([NZ]). With notation as above, for each $g \geq 0$ the real GW-invariant satisfies
\[ \text{GW}_{g,B}^{X,\phi}(\mu_1, \ldots, \mu_l) = \sum_{0 \leq h \leq g} \text{C}_{h,B}(\frac{g-h}{2}) E_{h,B}(\mu_1, \ldots, \mu_l), \]
where $\text{C}_{h,B}(\frac{g-h}{2})$ satisfies the generating function
\[ \sum_{g=0}^{\infty} \text{C}_{h,B}(g) t^{2g} = \left( \frac{\sinh(t/2)}{t/2} \right)^{h-1+c_1(B)/2}. \]

In [NZ], we compute the real GW-invariants for $\mathbb{P}^3$ with conjugate pairs of point constraints up to $g \leq 5$ and $d \leq 8$ by equivariant localization and transform them into the signed integer counts. These integers provide non-trivial lower bounds for counts of real curves in $\mathbb{P}^3$. The genus 0 numbers coincide with Welschinger’s invariants [We2].

We also obtain some implications for certain Hodge integrals from Theorem 5. For example, let $\mathcal{E}$ and $\psi_1$ be the Hodge bundle and the first Chern class of the universal cotangent bundle over $\overline{\mathcal{M}}_{g,1}$, respectively. Then for arbitrary formal variables $x$ and $y$
\[ 1 + \sum_{g=1}^{\infty} t^{2g} \int_{\overline{\mathcal{M}}_{g,1}} \frac{\Lambda(x+y)\Lambda(x)\Lambda(y)}{(x+y)(x+y-\psi_1)} = \frac{\sin(t/2)}{t/2}, \]
where $\Lambda(x) = \sum_{r=0}^{g} (-1)^r c_r(\mathcal{E}) x^{g-r}$. The specialization $y = -2x$ is equivalent to the $k = -2$ case of [FP] Theorem 2.

Further research. A long term goal of my research is to establish the (genus 1) mirror symmetry for real GW-invariants. Parallel to the approach described in Section 1, it should be possible to define the reduced real GW-invariants and the real version of the hyperplane relation for the quintic 3-fold in the genus 1 case. In light of the localization computation in [PoZ], the genus 1 mirror symmetry for real GW-invariants of the quintic 3-fold would then be established.

References


[RaSW] D. Ranganathan, K. Santos-Parker, and J. Wise, *Moduli of stable maps in genus one & logarithmic geometry I*, math/1708.02359


