MODULI OF CURVES OF GENUS ONE WITH TWISTED FIELDS

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Abstract. We construct a smooth Artin stack parameterizing the stable weighted curves of genus one with twisted fields and prove that it is isomorphic to the blowup stack of the moduli of genus one weighted curves studied by Hu and Li. This leads to a geometric construction without using blowups, as well as a modular interpretation of the points, of Vakil-Zinger’s desingularization of the moduli of genus one stable maps to projective spaces. Our results should extend to resolutions of the moduli of stable maps of higher genera.

1. Introduction

Moduli problems are of central importance in algebraic geometry. Many moduli spaces possess arbitrary singularities [12]. Among them, the moduli $\mathcal{M}_g(P^n, d)$ of degree $d$ stable maps from genus $g$ nodal curves into projective spaces $P^n$ are particularly important. We aim to resolve the singularities of $\mathcal{M}_g(P^n, d)$. The resolution of singularity is arguably one of the hardest problems in algebraic geometry [3, 4, 9, 10].

The stable map moduli are smooth if $g=0$ and singular if $g \geq 1$ and $n \geq 2$. For $g=1$, a resolution was constructed by Vakil and Zinger [13], followed by an algebraic approach of Hu and Li [5]. The latter is achieved by constructing a canonical smooth blowup $\tilde{\mathcal{M}}_1^{wt}$ of the Artin stack $\mathcal{M}_1^{wt}$ of weighted nodal curves of genus one. The method of [5] was further developed in [7] to finally establish a resolution in the case of $g=2$. The resolution of [7] is achieved by constructing a canonical smooth blowup $\tilde{\mathcal{P}}_2$ of the relative Picard stack $\mathcal{P}_2$ of nodal curves of genus two.

In higher genus cases, the construction of a possible resolution of the stable map moduli may seem formidable. The constructions of the explicit resolutions in [13, 5, 7] rely on certain precise knowledge on the singularities of the moduli. For arbitrary genus, it calls for a more abstract and geometric approach. As advocated by the first author, every singular moduli space should admit a resolution which itself is also a moduli. Following this principle, we provide a modular meaning to the stack $\tilde{\mathcal{M}}_1^{wt}$, which in turn leads to a modular interpretation of the resolution $\tilde{\mathcal{M}}_1(P^n, d)$ of $\mathcal{M}_1(P^n, d)$. To be somewhat more informative, we interpret the blowup stack $\tilde{\mathcal{M}}_1^{wt}$ of [5] as a smooth algebraic stack of stable weighted nodal curves of genus one with twisted fields, and consequently, the resolution $\tilde{\mathcal{M}}_1(P^n, d)$ of $\mathcal{M}_1(P^n, d)$ as a Deligne-Mumford stack of genus one stable maps with twisted fields. The results in this paper are the first step to tackle the arbitrary genus case.

The main theorem of this paper is the following:

Theorem 1.1. There exits a smooth Artin stack $\mathcal{M}_1^{tf}$ parameterizing the weighted nodal curves of genus one with twisted fields, along with a universal family $C^{tf} \rightarrow \mathcal{M}_1^{tf}$ and a proper forgetful morphism $\varpi : \mathcal{M}_1^{tf} \rightarrow \mathcal{M}_1^{wt}$. Moreover, $\mathcal{M}_1^{tf}/\mathcal{M}_1^{wt}$ is isomorphic to the blowup stack $\tilde{\mathcal{M}}_1^{wt}/\mathcal{M}_1^{wt}$.

We construct the strata of $\mathcal{M}_1^{tf}$ and the forgetful map $\varpi$ in [2]; see (2.13). We then glue the strata of $\mathcal{M}_1^{tf}$ together using smooth charts in [3] and conclude that $\mathcal{M}_1^{tf}$ is a smooth Artin stack in Corollary 3.7. The universal family $C^{tf} \rightarrow \mathcal{M}_1^{tf}$ is described in Proposition 3.9. We finally show that $\mathcal{M}_1^{tf}/\mathcal{M}_1^{wt}$ is isomorphic to $\tilde{\mathcal{M}}_1^{wt}/\mathcal{M}_1^{wt}$ in Proposition 4.3, which implies the properness of $\varpi$. These results together establish Theorem 1.1.

We point out that there should exist a groupoid, represented by $\mathcal{M}_1^{tf}$, that sends any scheme $S$ to the set of the flat families of stable weighted nodal curves of genus 1 with twisted fields over $S$ as in (3.32); see Remark 3.10 for some details.
According to [5], the resolution \( \widehat{\mathcal{M}}_1(\mathbb{P}^n, d) \) of \( \mathcal{M}_1(\mathbb{P}^n, d) \) is given by
\[
\widehat{\mathcal{M}}_1(\mathbb{P}^n, d) = \mathcal{M}_1(\mathbb{P}^n, d) \times_{\mathcal{M}_1^\text{wt}} \mathcal{M}_1^\text{wt},
\]
where
\[
\mathcal{M}_1(\mathbb{P}^n, d) \to \mathcal{M}_1^\text{wt}, \quad [C, \mathbf{u}] \mapsto [C, c_1(\mathbf{u}^* \mathcal{O}_{\mathbb{P}^n}(1))]
\]
and \( \mathcal{M}_1^\text{wt} \to \mathcal{M}_1^\text{wt} \) is the canonical blowup. Analogously, we take
\[
\widehat{\mathcal{M}}_1^\text{tf}(\mathbb{P}^n, d) := \mathcal{M}_1(\mathbb{P}^n, d) \times_{\mathcal{M}_1^\text{tf}} \mathcal{M}_1^\text{tf},
\]
where \( \mathcal{M}_1^\text{tf} \to \mathcal{M}_1^\text{wt} \) is the forgetful morphism aforementioned. Theorem 1.1 then leads to the following conclusion immediately.

**Corollary 1.2.** \( \widehat{\mathcal{M}}_1^\text{tf}(\mathbb{P}^n, d) \) is a proper Deligne-Mumford stack and is isomorphic to \( \widehat{\mathcal{M}}_1(\mathbb{P}^n, d) \).

Via the above isomorphism and applying [5], one sees that the stack \( \widehat{\mathcal{M}}_1^\text{tf}(\mathbb{P}^n, d) \) provides a resolution of \( \mathcal{M}_1(\mathbb{P}^n, d) \). Nonetheless, without relating to \( \widehat{\mathcal{M}}_1(\mathbb{P}^n, d) \), we can directly prove the resolution property of \( \widehat{\mathcal{M}}_1^\text{tf}(\mathbb{P}^n, d) \) by investigating the local equations of \( \mathcal{M}_1(\mathbb{P}^n, d) \) in [5] and their pullbacks to \( \widehat{\mathcal{M}}_1^\text{tf}(\mathbb{P}^n, d) \). This circle of ideas will be useful to treat the cases of higher genera. Further explanation can be found in Remark 3.8.

The methods and ideas of this paper can actually be extended to many other moduli spaces. For example, in [8], we generalize the results in [2] and [3] and construct a smooth Artin stack \( \mathcal{M}_2^\text{tf} \) of genus 2 nodal curves with line bundles and twisted fields, along with a proper forgetful morphism \( \mathcal{M}_2^\text{tf} \to \mathcal{M}_2 \), such that
\[
\mathcal{M}_2^\text{tf}(\mathbb{P}^n, d) = \mathcal{M}_2(\mathbb{P}^n, d) \times_{\mathcal{M}_2^\text{tf}} \mathcal{M}_2^\text{tf} \to \mathcal{M}_2(\mathbb{P}^n, d)
\]
is a resolution that factors through the blowing-up resolution in [7]. Further, we expect they can be extended to arbitrary genus, as far as the existence of moduli of nodal curves with twisted fields is concerned. This is the main motivation of the current article.

In a related work [11], D. Ranganathan, K. Santos-Parker, and J. Wise provide a different modular perspective of \( \widehat{\mathcal{M}}_1(\mathbb{P}^n, d) \) using logarithmic geometry.

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**Convention.** The subscript “1” of the relevant stacks indicating the genus appears only in \([1]\) and will be omitted starting \([2]\) as we only deal with the genus 1 case in this paper. In particular, we will denote by
\[
\mathcal{M}^\text{wt} \quad \text{and} \quad \mathcal{M}^\text{tf}
\]
the aforementioned stacks \( \mathcal{M}_1^\text{wt} \) and \( \mathcal{M}_1^\text{tf} \), respectively.

2. Set-theoretic descriptions

In \([2.1]\) we discuss the combinatorics of the dual graphs of nodal curves and introduce the notion of the weighted level trees. They will be used to define \( \mathcal{M}^\text{tf} \) set-theoretically in \([2.2]\).

**2.1. Weighted level trees.** Let \( \gamma \) be a rooted tree, i.e. a connected finite graph of genus 0 along with a special vertex \( o \), called the root. The sets of the vertices and the edges of \( \gamma \) are denoted by
\[
\text{Ver}(\gamma) \quad \text{and} \quad \text{Edg}(\gamma),
\]
respectively. The set \( \text{Ver}(\gamma) \) is endowed with a partial order, called the tree order, so that \( v > v' \) if and only if \( v \neq v' \) and \( v \) belongs to a path between \( o \) and \( v' \). The root \( o \) is thus the unique maximal element of \( \text{Ver}(\gamma) \) with respect to the tree order.

For each \( e \in \text{Edg}(\gamma) \), we denote by \( v_e^+ \in \text{Ver}(\gamma) \) the endpoints of \( e \) such that \( v_e^+ > v_e^- \). Then, every vertex \( v \in \text{Ver}(\gamma) \setminus \{o\} \) corresponds to a unique
\[
e_v \in \text{Edg}(\gamma) \quad \text{satisfying} \quad v_{v_e^-} = v.
\]
The tree order on \( \text{Ver}(\gamma) \) induces a partial order on \( \text{Edg}(\gamma) \), still called the tree order, so that 
\[
eq \ell' \iff \nu^{-}_e \geq \nu^{+}_{e'}.
\]

We call a pair \( \tau = (\gamma, \textbf{w}) \) consisting of a rooted tree \( \gamma \) and a function 
\[
\textbf{w} : \text{Ver}(\gamma) \to \mathbb{Z}_{>0}
\]
a weighted tree. For such \( \tau \), we write \( \text{Ver}(\tau) = \text{Ver}(\gamma) \) and \( \text{Edg}(\tau) = \text{Edg}(\gamma) \). The set of all the weighted trees is denoted by \( \mathcal{T}_{\text{wt}} \).

We call a map \( \ell : \text{Ver}(\gamma) \to \mathbb{R}_{\leq 0} \) satisfying
\[
\ell^{-1}(0) = \{ o \} \quad \text{and} \quad \ell(v) > \ell(v') \quad \text{whenever} \quad v > v'
\]
a level map. For each \( i \in \ell(\text{Ver}(\gamma)) \setminus \{ 0 \} \), let
\[
i^i = \min \{ k \in \ell(\text{Ver}(\gamma)) : k > i \},
\]
i.e. the level \( i^2 \) is right “above” the level \( i \); see Figure 1. We remark that a rooted tree along with a level map is called a level graph with the root as the unique top level vertex in [1 §1.5].

**Definition 2.1.** We call the tuple
\[
t = (\gamma, \textbf{w} : \text{Ver}(\gamma) \to \mathbb{Z}_{>0}, \ell : \text{Ver}(\gamma) \to \mathbb{R}_{\leq 0})
\]
a weighted level tree if \( (\gamma, \textbf{w}) \in \mathcal{T}_{\text{wt}} \) and \( \ell \) is a level map.

For every weighted level tree \( t \) as above, we write \( \text{Ver}(t) = \text{Ver}(\gamma) \) and \( \text{Edg}(t) = \text{Edg}(\gamma) \). Set
\[
\textbf{m} = \textbf{m}(t) = \max \{ \ell(v) : v \in \text{Ver}(t), \textbf{w}(v) > 0 \} \quad (\leq 0),
\]
\[
\textbf{Edg}(t) = \{ e \in \text{Edg}(t) : \ell(v_{e}^{-}) > \textbf{m} \} \quad (\subset \text{Edg}(t)).
\]

For any two levels \( i, j \in \mathbb{R}_{\leq 0} \), we write
\[
(i, j)_t = \ell(\text{Ver}(t) \cap (i, j)), \quad [i, j]_t = \ell(\text{Ver}(t) \cap [i, j]).
\]

For every \( e \in \text{Edg}(t) \), let
\[
\ell(e) = \max \{ \ell(v_{e}^{-}), \textbf{m} \} \quad (\in [\textbf{m}, 0]_t).
\]

For each level \( i \in [\textbf{m}, 0]_t \), we set
\[
\mathcal{E}_i = \mathcal{E}_i(t) = \{ e \in \text{Edg}(t) : \ell(e) \leq i < \ell(v_{e}^{+}) \}.
\]

In other words, \( \mathcal{E}_i \) consists of all the edges crossing the gap between the levels \( i \) and \( i^2 \).

We remark that all the notions in the preceding paragraph depend on the weighted level tree \( t \), although we may hereafter omit \( t \) in any of such notions when the context is clear.

Every weighted level tree \( t \) determines a unique index set
\[
\mathbb{I}(t) = \mathbb{I}_+(t) \cup \mathbb{I}_-(t) \cup \mathbb{I}_m(t), \quad \text{where} \quad \mathbb{I}_+(t) = [\textbf{m}, 0]_t,
\]
\[
\mathbb{I}_m(t) = \{ e \in \text{Edg}(t) : \ell(v_{e}^{-}) < \textbf{m} \}, \quad \mathbb{I}_-(t) = (\text{Edg}(t) \setminus \text{Edg}(t)).
\]

The set \( \mathbb{I}_+(t) \) becomes empty if \( \textbf{m} = 0 \), i.e. the root \( o \) is positively weighted. As mentioned before, we may simply write
\[
\mathbb{I} = \mathbb{I}(t), \quad \mathbb{I}_\pm = \mathbb{I}_\pm(t), \quad \mathbb{I}_m = \mathbb{I}_m(t)
\]
when the context is clear.

For each \( I \subseteq \mathbb{I} \) (possibly empty), let
\[
I_m = I \cap \mathbb{I}_m, \quad I_\pm = I \cap \mathbb{I}_\pm.
\]

We construct a weighted level tree
\[
(\tau_{(I)}, \ell_{(I)}) = (\gamma_{(I)}, \textbf{w}_{(I)}, \ell_{(I)})
\]
as follows:
the rooted tree $\gamma(t_I)$ is obtained via the edge contraction

$$\pi(t_I) : \text{Ver}(\gamma) \to \text{Ver}(\gamma(t_I))$$

such that the set of the contracted edges is

$$\text{Edg}(t_I) \setminus \text{Edg}(t) = \{e \in (\text{Edg}(t) \setminus \mathbb{I}_m) \cup I_m : [\ell(e), \ell(v^+)] \cup I \subset I_+ \} \cup I_-;$$

the weight function $w(t_I)$ is given by

$$w(t_I) : \text{Ver}(\gamma(t_I)) \to \mathbb{Z}_{\geq 0}, \quad w(t_I)(v) = \sum_{v' \in \pi^{-1}(v)} w(v');$$

the level map $\ell(t_I)$ is such that for any $e \in \text{Edg}(\gamma(t_I)) \subset \text{Edg}(t)$,

$$\ell(t_I)(v^-) = \begin{cases} \min\{i \in \mathbb{I}_+ \setminus I_+ : i \geq \ell(v) \land \forall v \in \pi^{-1}(v^-)\} & \text{if } e \in (\text{Edg}(t) \setminus \mathbb{I}_m), \\ \min(\mathbb{I}_+ \setminus I_+) & \text{if } e \in \mathbb{I}_m, \\ \max\{\ell(v) : v \in \pi^{-1}(v^-)\} & \text{if } e \in (\mathbb{I}_m \setminus \mathbb{I}_m) \cup I_-.
\end{cases}$$

Figure 1. A weighted level tree with chosen $v_1, v_2,$ and $v_3$

- the rooted tree $\gamma(t_I)$ is obtained via the edge contraction
- such that the set of the contracted edges is

$$(2.6) \quad \text{Edg}(t_I) \setminus \text{Edg}(t) = \{e \in (\text{Edg}(t) \setminus \mathbb{I}_m) \cup I_m : [\ell(e), \ell(v^+)] \cup I \subset I_+ \} \cup I_-;$$

- the weight function $w(t_I)$ is given by

$$w(t_I) : \text{Ver}(\gamma(t_I)) \to \mathbb{Z}_{\geq 0}, \quad w(t_I)(v) = \sum_{v' \in \pi^{-1}(v)} w(v');$$

- the level map $\ell(t_I)$ is such that for any $e \in \text{Edg}(\gamma(t_I)) \subset \text{Edg}(t)$,

$$\ell(t_I)(v^-) = \begin{cases} \min\{i \in \mathbb{I}_+ \setminus I_+ : i \geq \ell(v) \land \forall v \in \pi^{-1}(v^-)\} & \text{if } e \in (\text{Edg}(t) \setminus \mathbb{I}_m), \\ \min(\mathbb{I}_+ \setminus I_+) & \text{if } e \in \mathbb{I}_m, \\ \max\{\ell(v) : v \in \pi^{-1}(v^-)\} & \text{if } e \in (\mathbb{I}_m \setminus \mathbb{I}_m) \cup I_-.
\end{cases}$$

It is a direct check that $\pi(t_I)$ is a weighted tree and $\ell(t_I)$ satisfies the criteria of a level map, hence gives a well defined weighted level tree.

The construction of $t_I$ implies

$$\mathbb{I}_+(t_I) = \mathbb{I}_+ \setminus I_+, \quad \mathbb{m}(t_I) = \min((\mathbb{I}_+ \setminus I_+) \cup \{0\}),$$

$$(2.7) \quad \mathbb{I}_m(t_I) = \{e \in \mathbb{I}_m \setminus I_m : \ell(v^+)_e > \mathbb{m}(t_I)\},$$

$$\mathbb{I}_-(t_I) = \mathbb{I}_- \setminus I_- \cup \{e \in \mathbb{I}_m \setminus I_m : \ell(v^+)_e \leq \mathbb{m}(t_I)\}.$$ Intuitively, the weighted level tree $t_I$ is obtained from $t$ by contracting all the edges labeled by $I_-$, then lifting all the vertices $v$ with $e_v \in I_m$ to the level $m$, and finally contracting all the levels in $I_+$. Such $t_I$ will be used to describe the local structure of the stack $\mathcal{M}^{tf}$ in $[3].$

**Definition 2.2.** Two weighted level trees $t=(\gamma, w, \ell)$ and $t'=(\gamma', w', \ell')$ are said to be equivalent, written as $t \sim t'$, if

- (E1) $(\gamma, w) = (\gamma', w')$ as weighted trees;
- (E2) for any $v, w \in \text{Ver}(\gamma)$ satisfying $\ell(v) = \ell(w) = \ell_{m}(t)$, we have $\ell'(v) = \ell'(w)$;
- (E3) for any $v, w \in \text{Ver}(\gamma)$ satisfying $\ell(v) > \ell(w)$ and $\ell(w) \geq \mathbb{m}(t)$, we have $\ell'(v) > \ell'(w)$.

It is a direct check that $\sim$ is an equivalence relation on the set of weighted level trees. Intuitively, this equivalence relation records the relative positions of the vertices above or in the level $\mathbb{m}(t)$; see Figure 1 for illustration.
We denote by $\mathcal{S}_q^{\text{wt}}$ the set of the equivalence classes of the weighted level trees. There is a natural forgetful map
\begin{equation}
(2.8) \quad \phi : \mathcal{S}_q^{\text{wt}} \rightarrow \mathcal{S}_p^{\text{wt}}, \quad \phi([\gamma, \omega, \ell]) \rightarrow (\gamma, \omega),
\end{equation}
which is well defined by Condition (E1) of Definition 2.2.

If $t \sim t'$, then $m(t) = m(t')$, and there exists a bijection
\[ \phi_{\mathcal{E}^t} : \mathcal{P}(\emptyset(t)) \rightarrow \mathcal{P}(\emptyset(t)), \quad \phi_{\mathcal{E}^t}(I) = \mathcal{E}^t(\ell^{-1}(I_+)) \sqcup I_{m(t)} \sqcup I_-, \]
where $\mathcal{P}(\cdot)$ denotes the power set. The next lemma follows from direct check.

**Lemma 2.3.** If $t \sim t'$, then $t(I) \sim t'(\phi_{\mathcal{E}^t}(I))$ for any $I \in \emptyset(t)$.

### 2.2. Twisted fields.

For every genus 1 nodal curve $C$, its dual graph $\gamma_C^*$ has either a unique vertex $o$ corresponding to the genus 1 irreducible component of $C$ or a unique loop. In the former case, $\gamma_C^*$ can be considered as a rooted tree with the root $o$; in the latter case, we contract the loop to a single vertex $o$ and obtain a rooted tree with the root $o$. Such defined rooted tree is denoted by $\gamma_C$ and called the reduced dual tree of $C$ (c.f. [5 §3.4]). We call the minimal connected genus 1 subcurve of $C$ the core and denote it by $C_o$. Other irreducible components of $C$ are smooth rational curves and denoted by $C_v$, $v \in \text{Ver}(\gamma_C) \backslash \{o\}$. For every incident pair $(v, e)$, let
\begin{equation}
(2.9) \quad q_{v,e} \in C_v
\end{equation}
be the nodal point corresponding to the edge $e$.

Let $\mathcal{M}^{\text{wt}}$ be the Artin stack of genus 1 stable weighted curves introduced in [5 §2.1]. Here the subscript “1” indicating the genus is omitted as per our convention. The stack $\mathcal{M}^{\text{wt}}$ consists of the pairs $(C, \omega)$ of genus 1 nodal curves $C$ with non-negative weights $\omega \in H^2(C, \mathbb{Z})$, meaning that $\omega(\Sigma) \geq 0$ for all irreducible $\Sigma \subset C$. Here $(C, \omega)$ is said to be stable if every rational irreducible component of weight 0 contains at least three nodal points. The weight of the core $\omega(C_o)$ is defined as the sum of the weights of all irreducible components of the core.

Every $(C, \omega) \in \mathcal{M}^{\text{wt}}$ uniquely determines a function
\[ \omega : \text{Ver}(\gamma_C) \rightarrow \mathbb{Z}_{\geq 0}, \quad v \mapsto \omega(C_v), \]
which makes the pair $(\gamma_C, \omega)$ a weighted tree, called the weighted dual tree. Thus, the stack $\mathcal{M}^{\text{wt}}$ can be stratified as
\begin{equation}
(2.10) \quad \mathcal{M}^{\text{wt}} = \bigsqcup_{\tau \in \mathcal{S}_q^{\text{wt}}} \mathcal{M}_\tau^{\text{wt}} = \bigsqcup_{\tau \in \mathcal{S}_p^{\text{wt}}} \{ (C, \omega) \in \mathcal{M}^{\text{wt}} : (\gamma_C, \omega) = \tau \}.
\end{equation}

If the sum of the weights of all vertices is fixed, the stability condition of $\mathcal{M}^{\text{wt}}$ then guarantees there are only finitely many $\tau \in \mathcal{S}_q^{\text{wt}}$ so that $\mathcal{M}_\tau^{\text{wt}}$ is non-empty.

Given $\tau = (\gamma, \omega) \in \mathcal{S}_q^{\text{wt}}$ and $e \in \text{Edg}(\gamma)$, let
\[ L_e^{\pm} \rightarrow \mathcal{M}_\tau^{\text{wt}} \]
be the line bundles whose fibers over a weighted curve $(C, \omega)$ are the tangent vectors of the irreducible components $C_{v_{\pm}}$ at the nodal points $q_{v_{\pm}, e}$, respectively. We take
\begin{equation}
(2.11) \quad L_e = L_e^+ \otimes L_e^-, \quad L_e^\pm = \bigotimes_{e' \in \text{Edg}(\gamma), e' \geq e} L_{e'} \rightarrow \mathcal{M}_\tau^{\text{wt}}.
\end{equation}

For any direct sum of line bundles $V = \oplus_m L_m$ (over any base), we write
\[ \mathcal{P}(V) := \{ (x, [v_m]) \in \mathbb{P}V : v_m \neq 0 \forall m \} \]
For any morphisms $M_1, \ldots, M_k \rightarrow S$, we write
\[ \prod_{1 \leq i \leq k} (M_i/S) := M_1 \times_S M_2 \times_S \cdots \times_S M_k. \]
With notation as above, given $\tau \in \mathcal{S}_R^{\text{wt}}$ and $[t] = [\tau, \ell] \in \mathcal{S}_L^{\text{wt}}$, let
\[
\varpi: \mathcal{M}_{[t]}^{\text{tf}} = \left( \prod_{i \in \mathbb{N}_+ (t)} \left( \mathbb{P} \left( \bigoplus_{c \in \text{Edg}(t), \ell (v_c^\tau) = i} L_{c}^{\tau} \right) / \mathcal{M}_\tau^{\text{wt}} \right) \right) \rightarrow \mathcal{M}_\tau^{\text{wt}},
\]
(2.12)
\[
\mathcal{E}_{[t]} = \left( \prod_{i \in \mathbb{N}_+ (t)} \left( \mathbb{P} \left( \bigoplus_{c \in \mathbf{E}_i} L_{c}^{\tau} \right) / \mathcal{M}_\tau^{\text{wt}} \right) \right) \rightarrow \mathcal{M}_\tau^{\text{wt}},
\]
where $\mathbb{N}_+ (t)$, $L_{c}^{\tau}$, and $\mathbf{E}_i$ are as in (2.4), (2.11), and (2.3), respectively. It is straightforward that both bundles in (2.12) are independent of the choice of the weighted level tree $t$ representing $[t]$. Since
\[
\{ e \in \text{Edg}(t) : \ell (v_c^\tau) = i \} \subset \mathbf{E}_i, \quad \forall i \in \mathbb{N}_+ (t),
\]
we see that $\mathcal{M}_{[t]}^{\text{tf}}$ is a subset of $\mathcal{E}_{[t]}$. In addition, since each stratum $\mathcal{M}_\tau^{\text{wt}}$ is an algebraic stack, so are $\mathcal{M}_{[t]}^{\text{tf}}$ and $\mathcal{E}_{[t]}$.

Using (2.12) and (2.10), we define
\[
\mathcal{M}^{\text{tf}} := \bigsqcup_{[t] \in \mathcal{S}_L^{\text{wt}}} \mathcal{M}_{[t]}^{\text{tf}} \xrightarrow{\mathcal{M}_\tau^{\text{wt}}} \mathcal{M}_\tau^{\text{wt}}.
\]
(2.13)
This is the set-theoretic definition of the proposed stack $\mathcal{M}^{\text{tf}}$ as well as the forgetful map in Theorem 1.1. For any $x \in \mathcal{M}_\tau^{\text{wt}}$, the fibers of $\mathcal{M}^{\text{tf}}|_x$ are called the twisted fields over $x$.

**Remark 2.4.** By (2.13), $\mathcal{M}^{\text{tf}}_1 (\mathbb{P}^n, d)$ in Corollary 1.2 consists of the tuples
\[
(C, u, [t], \eta),
\]
where $(C, u)$ are stable maps in $\overline{M}_1 (\mathbb{P}^n, d)$, $[t]$ are the equivalence classes of weighted level trees satisfying $f[t] = (\gamma_C, c_1 (u^* \mathcal{O}_{\mathbb{P}^n} (1)))$, and $\eta$ are twisted fields over $(C, c_1 (u^* \mathcal{O}_{\mathbb{P}^n} (1)))$.

### 3. The Stack Structure of $\mathcal{M}^{\text{tf}}$

In [3], we show $\mathcal{M}^{\text{tf}}$ is naturally a smooth Artin stack and describe its universal family.

#### 3.1. Twisted charts

We first fix $[t] = [\gamma, w, \ell] \in \mathcal{S}_L^{\text{wt}}$ and $x \in \mathcal{M}_{[t]}^{\text{tf}}$, and write
\[
\tau = f[t] = (\gamma, w) \in \mathcal{S}_R^{\text{wt}}, \quad (C, w) = \varpi (x) \in \mathcal{M}_\tau^{\text{wt}}.
\]
Since $\mathcal{M}_\tau^{\text{wt}}$ is smooth, we take an affine smooth chart
\[
\mathcal{V} = \mathcal{V}_{\varpi (x)} \rightarrow \mathcal{M}_\tau^{\text{wt}}
\]
containing $(C, w)$.

As in [3 §4.3] and [7 §2.5], there exists a set of regular functions
\[
(3.1) \quad \zeta_e \in \Gamma (\mathcal{O}_{\mathcal{V}}) \quad \text{with} \quad e \in \text{Edg}(\gamma),
\]
called the modular parameters, so that for each $e \in \text{Edg}(\gamma)$, the locus
\[
Z_e = \{ \zeta_e = 0 \} \subset \mathcal{V}
\]
is the irreducible smooth Cartier divisor on $\mathcal{V}$ where the node labeled by $e$ is not smoothed. For any $I \subset \mathbb{I} = \mathbb{I} (t)$, let
\[
\mathcal{V}_{(I)}^{\varpi} := \{ \zeta_e \neq 0 : e' \in (\text{Edg}(\gamma) \setminus \text{Edg}(\gamma (I))) \} \subset \mathcal{V},
\]
\[
\mathcal{V}_{(I)}^{\eta} := \mathcal{V}_{(I)}^{\varpi} \cap \{ \zeta_e = 0 : e \in \text{Edg}(\gamma (I)) \} \subset \mathcal{V}_{(I)}^{\varpi} \subset \mathcal{V}.
\]
Then, $\mathcal{V}_{(I)}^{\varpi}$ is an open subset of $\mathcal{V}$, and
\[
(3.2) \quad \mathcal{V}_{(I)}^{\eta} \rightarrow \mathcal{M}_\tau^{\text{wt}}.
\]
is a smooth chart of the stratum \( \mathcal{M}^\text{wt}_t \). Rigorously, the sets \( I \) and \( I \) depend on the choice of the weighted level tree \( t \) representing \( \mathfrak{I} \), however, \( \mathcal{V}^0_{(I)} \) and \( \mathcal{V}^\tau_{(I)} \) are independent of such choice; c.f. Lemma 2.3.

Given a set of the modular parameters as in (3.1), we may extend it to a set of local parameters on \( \mathcal{V} \) centered at \( (C, w) \):

\[
(3.3) \quad \{ \zeta_e \}_{e \in \text{Edg}(\gamma)} \cup \{ s_{j} \}_{j \in J} \quad \text{with} \quad (C, w) = 0 := (0, \ldots, 0),
\]

where \( J \) is a finite set. We do not impose other conditions on \( c_j \).

For each \( e \in \text{Edg}(\gamma) \), we set

\[
(3.4) \quad \partial_{\zeta_e} := (d\zeta_e)^e \in \Gamma(\mathcal{V}; T\mathcal{M}^\text{wt}).
\]

**Lemma 3.1.** The restriction of \( \partial_{\zeta_e} \) to \( \mathcal{Z}_e (\subset \mathcal{V}) \) is a nowhere vanishing section of the line bundle \( L_e \) in (2.11); i.e. it gives a trivialization of \( L_e \to \mathcal{Z}_e \).

**Proof.** Since the restriction of \( d\zeta_e \) to \( \mathcal{Z}_e = \{ \zeta_e = 0 \} \) is identically zero, we observe that the restriction of \( \partial_{\zeta_e} \) to \( \mathcal{Z}_e \) is a nowhere vanishing section of the normal bundle of \( \mathcal{Z}_e \). It is a well-known fact of the moduli of curves that the normal bundle of \( \mathcal{Z}_e \) is \( L_e \); see [2, Proposition 3.31]. \( \square \)

For each level \( i \in \mathbb{I} = \mathbb{I}_+^I \), we choose a special vertex

\[
(3.5) \quad \nu_i \in \text{Ver}(\gamma) \quad \text{s.t.} \quad \ell(\nu_i) = i.
\]

We then denote by \( e_i, e_i^+ \), and \( \nu_i^+ \) respectively the edges and the vertex satisfying

\[
(3.6) \quad \nu[0] = i < \nu[1] = \ell(\nu_i^+) < \nu[2] = \ell(\nu_i^{+2}) < \cdots;
\]

see Figure 1 for illustration. This sequence is finite, as there is a unique step \( h \) satisfying \( \nu[h] = 0 \).

By Lemma 3.1, there exist \( \lambda_e \in \mathbb{A} \) with \( e \in \mathcal{E}^\text{Edg}(t) \) so that the fixed \( x \in \mathcal{M}_{[t]}^\text{END} \) over \( (C, w) \) can be uniquely written as

\[
(3.7) \quad x = \left( 0 : \prod_{i \in \mathbb{I}_+^I} \left[ \lambda_e \cdot \left( \bigotimes_{e \in e_i \in \mathbb{I}_+^I} \ell(e) = i \right) \right], \text{ where} \right.
\]

\[
\lambda_e \neq 0 \quad \forall \ e \in \mathcal{E}^\text{Edg}(t) \setminus \mathbb{I}_m, \quad \lambda_{e_i} = 1 \quad \forall \ i \in \mathbb{I}_+, \quad \lambda_e = 0 \quad \forall \ e \in \mathbb{I}_m.
\]

Let \( \mathcal{U}_x \subset \mathbb{A}_+^I \times \mathbb{A}_{\mathcal{E}^\text{Edg}(t) \setminus \mathbb{I}_m}^I \times \mathbb{A}_-^J \) be an open subset containing the point

\[
(3.8) \quad y_x := \left( 0, (\lambda_e)_{e \in \mathcal{E}^\text{Edg}(t) \setminus \mathbb{I}_m}, 0, 0 \right).
\]

The coordinates on \( \mathcal{U}_x \) are denoted by

\[
(3.9) \quad ((\varepsilon_i)_{i \in \mathbb{I}_+}, (u_e)_{e \in \mathcal{E}^\text{Edg}(t) \setminus \mathbb{I}_m}, (z_e)_{e \in \mathbb{I}_-}, (w_j)_{j \in J}).
\]

For any \( I \subset \mathbb{I} \), we set

\[
(3.10) \quad \mathcal{U}_x^I = \left\{ \varepsilon_i \neq 0 \ \forall \ i \in \mathbb{I}_+; \ \ u_e \neq 0 \ \forall \ e \in \mathbb{I}_m; \ \ z_e \neq 0 \ \forall \ e \in \mathbb{I}_- \right\} \subset \mathcal{U}_x,
\]

\[
\mathcal{U}_x^{[t], I} = \mathcal{U}_x^I \cap \left\{ \varepsilon_i = 0 \ \forall \ i \in \mathbb{I}_+ \setminus I_+; \ \ u_e = 0 \ \forall \ e \in \mathbb{I}_m \setminus I_m; \ \ z_e = 0 \ \forall \ e \in \mathbb{I}_- \setminus I_- \right\}.
\]

This gives rise to a stratification

\[
(3.11) \quad \mathcal{U}_x = \bigsqcup_{I \subset \mathbb{I}} \mathcal{U}_x^{[t], I}.
\]

We remark that neither \( \mathcal{U}_x \) nor its stratification (3.10) depends on the choice of the weighted level tree \( t \) representing \( \mathfrak{I} \), even though the sets \( \mathbb{I} \) and \( I \) depend on such choice. We also notice that \( \mathcal{U}_x^{[t], I} \) is an open subset of \( \mathcal{U}_x \), but the strata \( \mathcal{U}_x^{[t], I} \) are not open unless \( I = \mathbb{I} \).
For each $i \in \mathbb{I}_+$, we set

$$u_{e_i} = 1.$$  

By (3.7), shrinking $\mathfrak{U}_e$ if necessary, we have

$$u_e \in \Gamma(\mathcal{O}^*_{\mathfrak{U}_e}) \quad \forall \ e \in \mathcal{E}(t) \setminus \mathcal{I}_m.$$  

With the local parameters $\zeta_e$ and $\zeta_j$ as in (3.3), we construct a morphism

$$\theta_x : \mathfrak{U}_x \rightarrow \mathcal{V} \quad (\rightarrow \mathfrak{M}^{\mathfrak{wt}})$$

given by

$$\theta^*_x \zeta_e = \frac{u_{e_j} \cdot u_{e_{i+1}} \cdot u_{e_{i+1}}^* \cdots}{u_{e_{i+1}}^* \cdot u_{e_j} \cdot u_{e_{i+1}}^* \cdots} \prod_{i \in [\ell(e), \ell(e^*_j)])} \varepsilon_i \quad \forall \ e \in \mathcal{E}(t);$$

$$\theta^*_x \zeta_e = z_e \quad \forall \ e \in \mathbb{I}_- \setminus \mathcal{E}(t);$$

$$\theta^*_x \zeta_j = w_j \quad \forall \ j \in J.$$

The numerator and the denominator in the first line of (3.12) are both finite products, because (3.6) is always a finite sequence.

For any $I \subset \mathbb{I}$, it follows from (3.11) and (3.12) that

$$\theta_x(\mathfrak{U}_{x[I]}) \subset \mathcal{V}_{r[I]}, \quad \theta_x(\mathfrak{U}_{x[I+J]}) \subset \mathcal{V}_{r[I]},$$

where $\mathcal{V}_{r[I]}$ and $\mathcal{V}_{r[I]}$ are described before (3.2).

Fix $I \subset \mathbb{I}$ ($I$ may be empty). With

$$[\bar{r}(I)] = [\bar{r}(I), \ell(I)] \in \mathcal{P}_{\mathfrak{wt}}$$

as in (2.5) and $\mathfrak{M}^{\mathfrak{mf}}_{[r(I)]}$ as in (2.13), let

$$\Phi_{x(I)} : \mathfrak{U}_{x[I+J]} \rightarrow \mathcal{V}_{r[I]} \times \mathfrak{M}^{\mathfrak{mf}}_{[r(I)]} \quad (\rightarrow \mathfrak{M}^{\mathfrak{mf}}_{[r(I)]})$$

be the morphism so that for any $y \in \mathfrak{U}_{x[I+J]}$,

$$\Phi_{x(I)}(y) = \left(\theta_x(y) \cdot \prod_{i \in \mathbb{I}_+ \setminus I_+} \left[\mathfrak{O}_{x[I]} \left(\frac{\partial \zeta_i}{\partial \theta_x} \right) \right] ; \ e \in \mathcal{E}_I \right),$$

where

$$\mu_{e;i;I} = \left(\frac{u_{e_j} \cdot u_{e_{i+1}} \cdot u_{e_{i+1}}^* \cdots}{u_{e_{i+1}}^* \cdot u_{e_j} \cdot u_{e_{i+1}}^* \cdots} \prod_{e \in e_i ; [\ell(e), \ell(e^*_j)]) \in I} \theta^*_x \zeta_e \prod_{e \in e_i ; [\ell(e), \ell(e^*_j)]) \in I} \varepsilon_i \right)$$

for all $i \in \mathbb{I}_+ \setminus I_+$ and $e \in \mathcal{E}_I$. Similar to (3.12), the products in the first pair of parentheses above are both finite products.

By (3.13), the description of $\mathcal{V}_{r[I]}$ above (3.2), and (2.6), we see that

$$\mu_{e;i;I} \in \Gamma(\mathcal{O}^*_{\mathfrak{U}_{x[I+J]}}) \quad \forall \ I \subset \mathbb{I}, \ i \in \mathbb{I}_+ \setminus I_+, \ e \in \mathcal{E}_I.$$

Moreover, by (3.11),

$$\mu_{e;i;I} \mid_{\mathfrak{U}_{x[I+J]}} \begin{cases} = 0 & \text{if } \ell(I)(v_e^-) < i, \\ \in \Gamma(\mathcal{O}^*_{\mathfrak{U}_{x[I+J]}}) & \text{if } \ell(I)(v_e^-) = i. \end{cases}$$

This, along with (3.13), Lemma 3.1 and (2.12), implies $\Phi_{x(I)}$ is well-defined.

The morphisms $\Phi_{x[I], I \subset \mathbb{I}}$, together determine

$$\Phi_{x} : \mathfrak{U}_x \rightarrow \mathfrak{M}^{\mathfrak{mf}}, \quad \Phi_{x}(y) = \Phi_{x[I]}(y) \quad \text{if } y \in \mathfrak{U}_{x[I+J]}.$$  

We remark that $\Phi_{x(I)}$ and $\Phi_{x}$ are also independent of the choice of the weighted level tree $t$ representing $[t]$. Moreover, we observe that

$$\Phi_{x}(y) = \Phi_{x(\bar{z})}(yz) = x,$$
where \( y_x \in \mathfrak{U}_x \) is given in (3.8).

A priori \( \Phi_x \) is just a map, for the set-theoretic definition (2.13) of \( \mathfrak{M}^e \) does not describe its stack structure, although each \( \mathfrak{M}^e_{[t]} \) is a stack. In \( \text{§}3.2 \) we will show such \( \Phi_x \) patch together to endow \( \mathfrak{M}^e \) with a smooth stack structure. Each \( \Phi_x \) will hereafter be called a twisted chart centered at \( x \) (lying over \( V \to \mathfrak{M}^e \)), although rigorously it becomes a chart of \( \mathfrak{M}^e \) only after Corollary 3.7 is established.

**Lemma 3.2.** For any \( I \subset \mathbb{I} \), \( \Phi_{x(I)} : \mathfrak{U}_{x(I)} \to \mathfrak{M}^e_{[x(I)]} \) of (3.14) is an isomorphism to an open subset of \( \mathfrak{M}^e_{[x(I)]} \).

**Proof.** For any \( i \in I \), \( \mathfrak{U}_{x(I)} = \mathfrak{U}_{x(I)} \setminus I_+ \), notice that every edge in \( \mathcal{E}_i \) of the weighted level tree \( t \) is not contracted in the construction of \( t(x) \) (c.f. (2.6)). Thus,

\[
\mathcal{E}_x \subset \text{Edg}(t_x) \quad \forall \ i \in I(x).
\]

In particular, the edges \( e_i, i \in I_+(t_x) \), can be used as the special edges of \( t_x \). For conciseness, let

\[
\mathbf{E}[t_x] := \text{Edg}(t_x) \setminus \{ e_i : i \in I_+(t_x) \} \cup \mathbb{I}_m(t_x)
\]

be a set of the local parameters on \( V \) centered at \( x \) as in (3.3). By the definition of \( V_x \) above (3.2),

\[
(3.16) \quad \{ \zeta_e \}_{e \in \text{Edg}(t_x)} \cup \{ \zeta_j \}_{j \in J}
\]

is a set of local parameters of \( V_x \).

Recall that there exist \( \lambda_e \in \mathbb{A}^* \), \( e \in \text{Edg}(t_x) \), such that

\[
x = \left( 0 ; \prod_{i \in I_+} \left[ \lambda_e \cdot \left( \bigotimes_{e \in e_i} \partial_{\zeta_e} \big|_{\ell} : \ell(v_e) = i \right) \right] \right)
\]

as in (3.7). Let \( U_x : \mathbf{E}[t_x] \) be an open subset of \( (\mathbb{A}^*)^{\mathbf{E}[t_x]} \) such that

\[
(\lambda_e)_{e \in \mathbf{E}[t_x]} \in U_x : \mathbf{E}[t_x] \subset (\mathbb{A}^*)^{\mathbf{E}[t_x]}
\]

The coordinates of \( U_x : \mathbf{E}[t_x] \) are denoted by

\[
(\mu_e)_{e \in \mathbf{E}[t_x]}.
\]

In addition, we set

\[
\mu_e = 1 \quad \forall \ e \in I_+(t_x), \quad \mu_e = 0 \quad \forall \ e \in \mathbb{I}_m(t_x)
\]

Thus, the function \( \mu_e \) is defined for all \( e \in \text{Edg}(t_x) \), and is nowhere vanishing on \( U_x : \mathbf{E}[t_x] \) for all \( e \in \text{Edg}(t_x) \setminus \mathbb{I}_m(t_x) \).

The smooth chart \( \mathcal{V}_x : \mathcal{M}^e_{[x(I)]} \xrightarrow{\mathcal{M}^e_{[x(I)]}} \mathcal{M}^e_{[x(I)]} \) induces a smooth chart

\[
\mathcal{U}_x : \mathcal{U}[x(I)] := \mathcal{V}_x x \times U_x : \mathbf{E}[x(I)] \longrightarrow \mathcal{M}^e_{[x(I)]}
\]

given by

\[
(3.17) \quad \Psi_x : \mathcal{U}_x \longrightarrow \mathfrak{U}_{x(I)}
\]

We will construct a morphism

\[
(3.17) \quad \Psi_x : \mathcal{U}_x \longrightarrow \mathfrak{U}_{x(I)}
\]
such that $\Phi_{x(t)} \circ \Psi_{x(t)}$ and $\Psi_{x(t)} \circ \Phi_{x(t)}$ are both the identity morphisms, which will then establish Lemma 3.2.

Given $((i, (\mu_e)_{e \in E(t)}) \in U_{x(t)}^t$, we denote its image by

$$y := \Psi_{x(t)}((i, (\mu_e)_{e \in E(t)}) \in U_{x(t)}^t,$$

which is to be constructed. With the coordinates on $U_x$ as in (3.9), we set

$$z_e(y) = \zeta_e(i) \quad \forall e \in I_-, \quad w_j(y) = \zeta_j(i) \quad \forall j \in J.$$ 

By (3.2), we see that

$$z_e(y) = \zeta_e(i) = 0 \iff e \in I_-(t) \in (\subset I_-(t)).$$

The rest of the coordinates of $y$ are much more complicated; we describe them by induction over the levels in $I_+ = I_+(t)$. More precisely, we will show that $\varepsilon_i(y)$ with $i \in I_+$ and $u_e(y)$ with $e \in E_{\text{Edg}}(t) \setminus \{e_i : i \in I_+\}$ are all rational functions in $\zeta_e(i)$ and $\mu_{e^r}$, satisfying

$$\{\varepsilon_i(y) = 0 \iff i \in I_+(t)\} \quad \text{and} \quad \{u_e(y) = 0 \iff e \in I_m \setminus I_m\}.$$ 

In particular, (3.20) and (3.19) imply $y \in U_{x(t)}$, i.e. $\Psi_{x(t)}$ is well-defined.

The base case of the induction is for the level

$$i_0 := \max I_+(t).$$

We take

$$\varepsilon_{i_0}(y) = \zeta_{e_{i_0}}(i).$$

By (3.2), we see that $\varepsilon_{i_0}(y)$ satisfies (3.20). We take

$$u_{e_{i_0}}(y) = 1.$$

For any $e \neq e_{i_0}$ with $\ell(e) = i_0$, we set

$$u_e(y) = \begin{cases} 
\mu_e & \text{if } i_0 \notin I_+(t) \text{ (i.e. } i_0 \in I_-(t)\), \quad e \in E(t) \\
\zeta_e(i) & \text{if } i_0 \in I_+(t) \text{ (i.e. } i_0 \in E_{\text{Edg}}(t)\). 
\end{cases}$$

If $i_0 \in I_+$, then by (3.16) and (3.2), we have $\zeta_e(i) = 0$ if and only if $(m = i_0)$ and $e \in I_m \setminus I_m$. If $i_0 \notin I_+$, then $\mu_e = 0$ if and only if $(m = i_0)$ and $e \in I_m \setminus I_m$. We thus conclude that $u_e(y)$ satisfies (3.20) for all $e \in E_{\text{Edg}}(t)$ with $\ell(e) = i_0$. Moreover, such $\varepsilon_{i_0}(y)$ and $u_e(y)$ are obviously rational functions in $\zeta_e(i)$ and $\mu_{e^r}$. Hence, the base case is complete.

Next, for any $i \in I_+$, assume that all $\varepsilon_k(y)$ with $k > i$ and all $u_e(y)$ with $e \in E_{\text{Edg}}(t)$ and $\ell(e) > i$ have been expressed as rational functions in $\zeta_e(i)$ and $\mu_{e^r}$, satisfying (3.20).

For the level $i$, we first construct $\varepsilon_i(y)$. The construction is subdivided into three cases.

**Case 1.** If $i \notin I_+$, then set

$$\varepsilon_i(y) = 0.$$ 

Obviously this satisfies (3.20).

**Case 2.** If $i \in I_+$ and $[i, [i+1]] \notin I_+$, then $e_i \in E(t_i)$, hence $\mu_{e_i} \neq 0$. Let

$$\hat{i} := \min ([i, [i+1]] \setminus I_+) \in I_+(t_i).$$

Intuitively, $\hat{i}$ is the level containing the image of $v_i$ in $t_i$. Thus,

$$\ell_i(e_i) = \hat{i}.$$ 

Let $\varepsilon_i(y)$ be given by

$$\mu_{e_i} = \varepsilon_i(y) \cdot u_{e_i}^1(y) \cdot u_{e_i}^2(y) \cdots \prod_{e \geq e_i, \ell(e) \notin I_+} \zeta_e(i) \cdot \prod_{e \geq e_i, \ell(e) \notin I_+} \zeta_e(i).$$

(3.23)
The inductive assumption implies that all \( \varepsilon_h(y) \) with \( h \in [i, \hat{i}]_1 \), as well as all \( u_{e_\hat{i}}^+(y) \) and \( u_{e_i}^+(y) \) with \( h \geq 0 \), are non-zero and are rational functions in \( \zeta_{\ell}(\hat{z}) \) and \( \mu_{e\hat{r}} \). By (3.16) and (2.6), we also see that all \( \zeta_{\ell}(\hat{z}) \) and \( \zeta_{\ell}(\hat{z}) \) in (3.23) are non-zero. Therefore, such defined \( \varepsilon_i(y) \) is a rational function in \( \zeta_{\ell}(\hat{z}) \) and \( \mu_{e\ell} \), satisfying (3.20).

**Case 3.** If \([i, i[1]) \subset I_+ \) (hence \( i \in I_+ \)), then we see \( \varepsilon_i \in \mathrm{Edg}(t(I)) \) \( \setminus \text{Edg}(t(I)) \); c.f. (2.6). Intuitively, this means \( \varepsilon_{i} \) is contracted in the construction of \( t(I) \). By the description of \( \mathcal{V}(I) \) above (3.2), we see that \( \zeta_{\ell}(\hat{z}) \neq 0 \). Let \( \varepsilon_i(y) \) be given by

\[
\zeta_{e_i}(\hat{z}) = \varepsilon_i(y) \cdot \prod_{h \in [i, \hat{i}]_1} \varepsilon_h(y).
\]

Mimicking the argument in Case 2, we conclude that \( \varepsilon_i(y) \) is a rational function in \( \zeta_{\ell}(\hat{z}) \) and \( \mu_{e\ell} \), satisfying (3.20).

Next, we construct \( u_e(y) \) for \( e \in \overrightarrow{\text{Edg}}(t) \) with \( \ell(e) = i \). Set

\[
u_e(y) = 1.
\]

For \( e \neq e_i \), the construction is subdivided into two cases.

**Case A.** If \([i, \ell(e_i^+)] \subset I_+ \), then

\[
\overrightarrow{\varepsilon} \in \mathcal{E}(t(I)) \cup \mathbb{I}_m(t(I)) = \overrightarrow{\text{Edg}}(t(I)) \setminus \{ e_i : i \in \mathbb{I}_+(t(I)) \},
\]

hence \( \mu_e \) exists, and \( \mu_e = 0 \) if and only if \( e \in \mathbb{I}_m(t(I)) \). In Case A, since \([i, \ell(e_i^+)] \subset I_+ \), (2.7) further implies

\[
\mu_e = 0 \quad \iff \quad e \in \mathbb{I}_m \setminus \mathbb{I}_m.
\]

Let \( \kappa = \kappa_e := \min \{ [i, \ell(e_i^+)] \subset I_+ \} \). In Case A, \( \ell(I)(e) = \kappa \).

Let \( u_e(y) \) be given by

\[
\zeta_{e_i}(\hat{z}) = u_{\hat{i}}(y) \cdot \prod_{e > e_i, [\ell(e), \ell(e_i^+)] \subset I} \zeta_{\ell}(\hat{z}) \cdot \prod_{h \in [i, \hat{i}]_1} \varepsilon_h(y).
\]

We observe that if \( i \notin I_+ \), then \( \kappa = i \) and hence \( \prod_{h \in [i, \hat{i}]_1} \varepsilon_h(y) = 1 \); if \( i \in I_+ \), then the previous construction of \( \varepsilon_i(y) \), along with the inductive assumption, guarantees \( \prod_{h \in [i, \hat{i}]_1} \varepsilon_h(y) \neq 0 \). Mimicking the argument of Case 2 of the construction of \( \varepsilon_i(y) \) and taking (3.25) into account, we see that \( u_e(y) \) determined by (3.26) is a rational function in \( \zeta_{\ell}(\hat{z}) \) and \( \mu_{e\ell} \), and it satisfies (3.20).

**Case B.** If \([i, \ell(e_i^+)] \subset I_+, \) then (2.6) gives

\[
\varepsilon_i \in \overrightarrow{\text{Edg}}(t(I)) \quad (\text{i.e. } \zeta_{\ell}(\hat{z}) = 0) \quad \iff \quad e \in \mathbb{I}_m \setminus \mathbb{I}_m.
\]

Let \( u_e(y) \) be given by

\[
\zeta_{e_i}(\hat{z}) = u_{e_i}(y) \cdot \prod_{e = e_i, [\ell(e), \ell(e_i^+)] \subset I} \zeta_{\ell}(\hat{z}) \cdot \prod_{h \in [i, \hat{i}]_1} \varepsilon_h(y).
\]

Once again, mimicking the argument of Case A of the construction of \( u_e(y) \), and taking (3.27) as well as the description of \( \mathcal{V}(I) \) right before (3.2) into account, we see that \( u_e(y) \) determined by (3.28) is a rational function in \( \zeta_{\ell}(\hat{z}) \) and \( \mu_{e\ell} \), and it satisfies (3.20).

The cases 1-3, A, and B together complete the inductive construction of \( \Psi_{x(I)} \). Moreover, comparing

- (3.18) with the second line of (3.12),
- (3.21), the second case of (3.22), (3.24), and (3.28) with the first line of (3.12),
- the first case of (3.22), (3.23), and (3.26) with the expressions of \( \mu_{e;i\ell} \) right after (3.14),...
we observe that $\Psi_{x(t)}$ is the inverse of $\Phi_{x(t)}$.\hfill\Box

**Corollary 3.3.** $\Phi_x: \mathcal{U}_x \to \mathcal{M}_x^{sf}$ is injective.

**Proof.** This follows from Lemma 3.2 and the stratification (3.10) and (2.13) directly.\hfill\Box

### 3.2 Stack structure

In this subsection, we will show the twisted charts $\Phi_x$ patch together to endow $\mathcal{M}_x^{sf}$ with a smooth stack structure; c.f. Proposition 3.6 and Corollary 3.7. Note that a priori, $\Phi_x$ depends on the choices of the special vertices \{v_i\}_{i \in \mathbb{I}_+} (3.4) and of the local parameters (3.3). Nonetheless, Lemmas 3.4 and 3.5 below will guarantee that such choices do not affect the proposed stack structure of $\mathcal{M}_x^{sf}$, hence will make the proof of Proposition 3.6 more concise.

Let \{\nu_i: i \in \mathbb{I}_+\} be another set of the special vertices satisfying (3.4), and $\nu_i^+, d_i, d_i^+$ be the analogues of $v_i^+, e_i, e_i^+$ in (3.5), respectively. As in (3.6), each level $i \in \mathbb{I}_+$ similarly determines a finite sequence

$$i(0) = i < i(1) = \ell(\nu_i^+) < i(2) = \ell(\nu_i^+) < \cdots.$$ We take an open subset $\mathcal{U}_x' \subset A_x^{\mathbb{I}_+} \times A_\text{Edg}(t) \setminus \{d_i: i \in \mathbb{I}_+\} \times A_\ell \times A_J$

with the coordinates

$$((\delta_i)_{i \in \mathbb{I}_+}, (u_i^+)_{a \in \text{Edg}(t) \setminus \{d_i: i \in \mathbb{I}_+\}}, (\varepsilon_i)_{e \in \mathbb{I}_-}, (w_i^+)_{j \in J}).$$

as in (3.9), and then construct

$$\theta_i^a: \mathcal{U}_x' \to \mathcal{V}_i, \quad \mu_i^a: I \subset \Gamma(\mathcal{V}_i), \quad \Phi_i^a: \mathcal{U}_x' \to \mathcal{M}_x^{sf}$$

parallel to (3.12) and (3.14). Let $\mathcal{U} = \Phi_i^a(\mathcal{U}_x') \cap \Phi_x(\mathcal{U}_x)$.

**Lemma 3.4.** The transition map

$$(\Phi_i^a)^{-1} \circ \Phi_x : \Phi_x^{-1}(\mathcal{U}) \to (\Phi_i^a)^{-1}(\mathcal{U})$$

is an isomorphism.

**Proof.** Let $g: \Phi_x^{-1}(\mathcal{U}) \to (\Phi_i^a)^{-1}(\mathcal{U})$ be the isomorphism given by

$$g^* \delta_i = \varepsilon_i, \quad u_i^+ = u_i^+, \quad \cdots, \quad u_i^+, u_i^+, \cdots, u_i^+, u_i^+, \cdots, \quad \forall i \in \mathbb{I}_+,$$

$$g^* u_i^e = \frac{u_i^e}{u_i^+} \quad \forall e \in \text{Edg}(t) \setminus \{d_i: i \in \mathbb{I}_+\}, \quad g^* z_e^+ = z_e \quad \forall e \in \mathbb{I}_-, \quad g^* w_j^+ = w_j \quad \forall j \in J.$$

The fact that $g$ is an isomorphism can be shown by constructing its inverse explicitly, which is similar to the proof of Lemma 3.2, but is simpler. The key point of the construction is that

$$u_i^+ = \frac{1}{g^* u_i^+} \quad \forall i \in \mathbb{I}_+, \quad u_e = \frac{g^* u_e}{g^* u_i^+} \quad \forall e \in \text{Edg}(t) \text{ with } \ell(e) = i,$$

and each $\varepsilon_i$ is a product of $g^* \delta_i$ and a rational function of $u_i^+$ with $e \in \text{Edg}(t)$.

It is a direct check that the isomorphism $g$ satisfies

$$\theta_i^a \circ g = \theta_x \quad \text{and} \quad g^* \mu_i^a = \frac{\mu_{i; e, I}}{\mu_{i; d, I}} \quad \forall I \subset \mathbb{I}, \quad i \in \mathbb{I}_+ \setminus I, \quad e \in \mathcal{E}_i.$$

Thus, $(\Phi_i^a)^{-1} \circ \Phi_x = g$ and hence is an isomorphism.\hfill\Box

Let

$$\{\zeta_e: e \in \text{Edg}(\gamma)\} \cup \{\zeta_j: j \in J\}$$

be another set of extended modular parameters centered at $x$ on the same chart $\mathcal{V} \to \mathcal{M}_x^{s}$. We use this set of local parameters to construct another twisted chart $\hat{\Phi}_x: \hat{\mathcal{U}}_x \to \mathcal{M}_x^{sf}$; in particular,
we have $\hat{\theta}_x : \hat{U}_x \to \mathcal{V}$ and $\hat{\mu}_{e;i; t} \in \Gamma(\mathcal{O}_{\hat{U}_x})$ as in (3.12) and (3.14), respectively. Parallel to (3.9), the coordinates on $\hat{U}_x$ are denoted by

$$((\hat{z}_i)_{i \in \mathbb{N}_+}, (\hat{v}_e)_{e \in \text{Edg}(t)}, (\hat{w}_j)_{j \in J}).$$

Let $\mathcal{U} = \Phi_x(U_x) \cap \Phi_x(\hat{U}_x)$.

**Lemma 3.5.** The transition map

$$(\Phi_x)^{-1} \circ \Phi_x^{-1} : \mathcal{U} \to \Phi_x^{-1}(\mathcal{U})$$

is an isomorphism.

**Proof.** By Lemma 3.4, it suffices to use the same set of the special vertices $\{v_i\}_{i \in \mathbb{N}_+}$ for both $\Phi_x$ and $\hat{\Phi}_x$. For any $e \in \text{Edg}(\gamma)$, the local parameters $\hat{z}_e$ and $z_e$ defines the same locus $Z_e = \{z_e = 0\} = \{\hat{z}_e = 0\}$, hence there exists $f_e \in \Gamma(\mathcal{O}_x)$ such that

$$\hat{z}_e = f_e \cdot z_e.$$

Therefore, we have

(3.29)

$$\partial_{z_e} \big|_{z_e} = \left( \frac{\partial}{\partial f_e} \right)^* \partial_{z_e} \big|_{z_e}.$$

Let $g : \Phi_x^{-1}(\mathcal{U}) \to \Phi_x^{-1}(\mathcal{U})$ be the isomorphism given by

$$g^* \hat{z}_e = \hat{z}_e \big|_{z_e} \cdot \frac{f_e}{f_e}; \quad \forall e \in \mathbb{N}_+, \quad g^* \hat{z}_e = z_e \big|_{z_e} \quad \forall e \in \mathbb{N}_-, \quad g^* \hat{w}_j = w_j \quad \forall j \in J,$$

$$g^* \hat{v}_e = v_e \big|_{z_e} \cdot \prod_{e' \in \mathcal{E}(e)} f_{e'} \quad \forall e \in \text{Edg}(t) \setminus \{e_i : i \in \mathbb{N}_+\}.$$

The explicit expression of $g$ above implies it is invertible; see the parallel argument in the proof of Lemma 3.4.

It is a direct check that $\hat{\theta}_x \circ g = \theta_x$ and

$$g^* \hat{\mu}_{e;i; t} = \hat{\mu}_{e;i; t} \big|_{z_e} \big|_{z_e} \cdot \frac{\mu_{e;i; t}}{\prod_{e \in \mathcal{E}(e)} f_{e'}} \quad \forall I \subset \mathbb{N}, \quad i \in \mathbb{N}_+ \setminus I_+, \quad e \in \mathcal{E}_i.$$

Taking (3.29) into account, we conclude that $(\Phi_x)^{-1} \circ \Phi_x = g$ and hence is an isomorphism.

Given $I \subset \mathbb{N}$ and $x' \in \Phi_x(U_x[t_i])$, let $\mathcal{V}_x(x') \to \mathcal{V}$ be a chart containing $x(x')$ and $\Phi_{x'} : \mathcal{V}_x \to \mathcal{V}$ is a twisted chart centered at $x'$ over $\mathcal{V}_x(x') \to \mathcal{V}_{x'}$. Let $\mathcal{U} = \Phi_x(U_x) \cap \Phi_{x'}(U_{x'})$.

**Proposition 3.6.** The transition map

$$\Phi_{x'}^{-1} \circ \Phi_x^{-1} : \mathcal{U} \to \Phi_{x'}^{-1}$$

is an isomorphism.

**Proof.** Since $x' \in \Phi_x(U_x[t_i]) \subset \Phi_x(U_x)$, its underlying weighted curve satisfies

$$\varpi(x') \in \mathcal{V}_x^0$$

Thus, replacing $\mathcal{V}_x(x')$ by $\mathcal{V}_x(x') \cap \mathcal{V}_x^0$ if necessary, we may assume

$$\varpi(x') \in \mathcal{V}_{x'}(x) \subset \mathcal{V}_{x'}^0.$$

Moreover, the following modular parameters on $\mathcal{V}$:

$$\zeta_e, \quad e \in \text{Edg}(t_i) \subset \text{Edg}(t)$$

also serve as modular parameters on $\mathcal{V}_{x'}(x')$. Thus by Lemma 3.5, we may assume $\Phi_{x'}$ is constructed using the local parameters on $\mathcal{V}_{x'}(x')$:

$$\{\zeta_e \in \text{Edg}(t_i) \} \cup \{\zeta_j \in J \cup \{\zeta_e \in \text{Edg}(t) \setminus \text{Edg}(t_i)\}_{i \in \mathbb{N}_+}\}.$$
as the analogue of (3.3).

Let the special vertices $v_i$ and edges $e_i$ of $t$ be respectively as in (3.4) and (3.5). By Lemma 3.4 we may further assume that the special vertices and edges of $t_i$ are respectively

$$v_i, e_i, i \in \mathbb{I}_+(t_i) = \mathbb{I}_+ \cap I_+.$$

For any $i \in \mathbb{I}_+ \cap I_+$ and $\hat{h} \in \mathbb{Z}_{\geq 0}$, let $i^+, e_i^+, \hat{h}(e_i)$ be the analogues of $i^*, e_i^*$, and $\hat{h}$, respectively, for the weighted level tree $t_i$ instead of $t$; see (2.1), (3.5), and (3.6) for notation.

Recall that $\mathbb{I}_-(t_j) = \mathbb{I}_- \cap J_+ \{ e \in \mathbb{I}_m : \ell(e) \leq \mathbf{m}(t_j) \}$. We denote by

$$((\varepsilon_i^j)_{e \in \mathbb{I}_+ \cap I_+}, (\zeta_i^j)_{e \in \mathbb{I}_- \cap (t_j)}, (\xi^j)_{j \in J \cup \{ \text{Edg}(t_i) \}})$$

the coordinates on $\mathcal{U}_{\mathcal{P}}$ parallel to (3.9), and construct

$$\theta_{x'} : \mathcal{U}_{x'} \to \mathcal{V}_{\mathcal{P}(x')}$$

and

$$\mu_{x;i'} \in \Gamma(\mathcal{O}_{\mathcal{U}_{x',i'}}, I') \subset I, \ i \in \mathbb{I}_+, \ x \in \mathcal{E}_i,$

parallel to

$$\theta_x : \mathcal{U}_x \to \mathcal{V}$$

and

$$\mu_{x;i} \in \Gamma(\mathcal{O}_{\mathcal{U}_{x,i}}, I), \ i \in \mathbb{I}_+ \cap I_+, \ x \in \mathcal{E}_i.$$

of (3.12) and (3.14), respectively. In this way, $\Phi_{x'} : \mathcal{U}_{x'} \to \mathcal{M}_{\text{aff}}$ is constructed analogously to $\Phi_x$.

Let $g : \Phi_{x'}^{-1}(U) \to \Phi_x^{-1}(U)$ be the isomorphism given by

$$g^* \varepsilon_i = \varepsilon_i \cdot \left( \prod_{j \in \mathcal{I}_+(t_j)} \varepsilon_{i_j} \cdot \prod_{e \in \mathcal{E}_j \cap I_j} \theta_{x_j} \varepsilon_e \cdot \frac{u_{e_i^j} \cdot u_{e_i^j} \cdots}{u_{e_i^j} \cdot u_{e_i^j} \cdots} \right) \prod_{e \in \mathcal{E}_j \cap I_j} \theta_{x_j} \varepsilon_e \cdot \frac{u_{e_j^j} \cdot u_{e_j^j} \cdots}{u_{e_j^j} \cdot u_{e_j^j} \cdots} \quad \forall \ i \in \mathbb{I}_+ \cap I_+$$

and

$$g^* \varepsilon_i = \varepsilon_i \cdot \left( \prod_{j \in \mathcal{I}_+(t_j)} \varepsilon_{i_j} \cdot \prod_{e \in \mathcal{E}_j \cap I_j} \theta_{x_j} \varepsilon_e \cdot \frac{u_{e_i^j} \cdot u_{e_i^j} \cdots}{u_{e_i^j} \cdot u_{e_i^j} \cdots} \right) \prod_{e \in \mathcal{E}_j \cap I_j} \theta_{x_j} \varepsilon_e \cdot \frac{u_{e_j^j} \cdot u_{e_j^j} \cdots}{u_{e_j^j} \cdot u_{e_j^j} \cdots} \quad \forall \ i \in \mathbb{I}_+ \cap I_+.$$

To see $g$ is well defined, notice that $\mathcal{V}_{\mathcal{P}(x')} \subset \mathcal{V}_{\mathcal{P}(1)}$ implies that

$$\Phi_{x'}^{-1}(U) \subset \mathcal{U}_{x',i'}.$$

Thus, every $\mu_{x;i}$ above can be considered as a function on $\Phi_{x'}^{-1}(U)$. By (3.30) and (3.13), the function $\theta_{x} \varepsilon_e$ with $e \in \mathcal{E}(t) \cup \mathcal{M}$ is nowhere vanishing on $\Phi_{x'}^{-1}(U)$ whenever $((\ell(e), \ell(e^*))) \subset I$. Taking (3.11) and (3.15) into account, we conclude that $g$ is well defined.

Once again, the explicit expression of $g$ implies it is invertible; see the parallel argument in the proof of Lemma 3.4. Moreover, it is a direct check that $\theta_{x'} \circ g = \theta_x$ and

$$g^* \mu_{x;i',i'} = \mu_{x;i \cup I'} \in \Gamma(\mathcal{O}_{\mathcal{U}_{x',i'}}, \Phi_x^{-1}(U)) \quad \forall \ i' \subset I, \ i \in \mathbb{I}_+(I_+ \cup I'), \ e \in \mathcal{E}_i.$$

Thus, $\Phi_{x'}^{-1} \circ \Phi_x = g$ and hence is an isomorphism.

\[\square\]

**Corollary 3.7.** \(\mathcal{M}_{\text{aff}}\) is a smooth Artin stack with \(\{ \Phi_x : \mathcal{U}_x \to \mathcal{M}_{\text{aff}} \}_{x \in \mathcal{M}^w} \) as smooth charts. Moreover, the structure of the stratification (2.13) is locally identical to the one induced by (3.10). Furthermore, for any \(x \in \mathcal{M}_{\text{aff}}\), any chart \(\mathcal{V} \to \mathcal{M}^w\) containing \(\mathcal{P}(x) \in \mathcal{M}^w\), and any twisted chart \(\Phi_x : \mathcal{U}_x \to \mathcal{M}_{\text{aff}}\) centered at \(x\) lying over \(\mathcal{V} \to \mathcal{M}^w\), we have

$$\begin{array}{ccc}
\mathcal{U}_x & \xrightarrow{\theta_x} & \mathcal{M}_{\text{aff}} \\
\Phi_x & \downarrow \Phi_x & \downarrow \mathcal{V} \\
\mathcal{V} & \to & \mathcal{M}^w
\end{array}$$

where \(\mathcal{V}\) is the forgetful morphism as in (2.13) and \(\theta_x\) is as in (3.12).
We may take the coordinates of $U$ be an open subset containing the point $p$ centered at $\theta$.

Remark 3.8. By (3.12) and Corollary 3.7, one sees that on an arbitrary twisted chart $U_\xi$ of $M_\text{tf}$,

$$\varpi^a \left( \prod_{\ell \geq e_2} \zeta_\ell \right) = \left( u_{e_2} + u_{e_3} + \cdots \right) \prod_{i \in I_{m+1}} \zeta_i,$$

$$\varpi^a \left( \prod_{\ell \geq e_2} \zeta_\ell \right) = \left( u_{e_2} u_{e_3} + \cdots \right) \prod_{i \in I_{m+1}} \zeta_i = u_{e_2} \cdot \varpi^a \left( \prod_{\ell \geq e_2} \zeta_\ell \right) \quad \forall \, e \in E_m.$$  

This, along with the local equations of $M_1(P^n, d)$ in [5], implies that the primary component of $M_1(P^n, d)$ is smooth and $M_1(P^n, d)$ contains at worst normal crossing singularities. This observation should be useful for the cases of higher genera.

3.3. A simple example. Let $[t] = [\gamma, w, \ell] \in S_{L}^\text{wt}$ be given by the leftmost diagram in Figure 2. Let $x \in M_\text{tf}$ be a weighted curve of genus 1 with twisted fields over $(C, w) \in M_\text{wt}$. The core and the nodes of $C$ are labeled by $o$ and by $a, b, c, d$, respectively.

Let $V \to M_\text{wt}$ be an affine smooth chart containing $(C, w)$, with a set of local parameters

$$\{ \zeta_a, \zeta_b, \zeta_c, \zeta_d \}$$

centered at $(C, w)$, where $\zeta_a, \ldots, \zeta_d$ are the modular parameters. There then exist non-zero $\lambda_c$ and $\lambda_d$ such that

$$x = \left( 0; [0, \partial_\zeta_a], [\partial_\zeta_b, \lambda_c \cdot (\partial_\zeta_a \otimes \partial_\zeta_c)[\zeta]], \lambda_d \cdot (\partial_\zeta_a \otimes \partial_\zeta_d)[\zeta] \right).$$

We choose the special edges (3.5) of $t$ to be $e_1 = b$ and $e_2 = a$. Let

$$\Omega_x \subseteq \mathbb{A}^{(-1, -2)} \times \mathbb{A}^{[c, d]} \times \mathbb{A}^{J}$$

be an open subset containing the point

$$y_x = (0, 0, \lambda_c, \lambda_d, 0, \ldots, 0).$$

The coordinates of $\Omega_x$ are denoted by

$$\varepsilon_{-1}, \varepsilon_{-2}, u_c, u_d, \quad \text{and} \quad w_j \text{ with } j \in J.$$  

We may take $\Omega_x = \{ u_c \neq 0, u_d \neq 0 \}$.

By Corollary 3.7 and (3.12), the forgetful morphism $\varpi : M_\text{tf} \to M_\text{wt}$ can locally be written as $\theta_x : \Omega_x \to V$ such that

$$\theta_x \left( \varepsilon_{-1}, \varepsilon_{-2}, u_c, u_d, (w_j) \right) = \left( \varepsilon_{-1} \cdot \varepsilon_{-2}, \varepsilon_{-1} \cdot \zeta_a, \varepsilon_{-2} \cdot u_c, \varepsilon_{-2} \cdot u_d, (w_j) \right).$$

Figure 2. Relevant weighted level trees in [3.3]
Considering all possible subsets \( I \) of \( \mathbf{I} \) in \( (3.14) \), we obtain a twisted chart \( \Phi_x: \Upsilon_x \to \mathcal{M}_t \) centered at \( x \) over \( \mathcal{V} \to \mathcal{M}_t \) so that for any
\[
y = (\varepsilon_1, \varepsilon_2, u_c, u_d, (w_j)) \in \Upsilon_x, \]
- if \( \varepsilon_{-1} = \varepsilon_{-2} = 0 \), then
  \[
  \Phi_x(y) = \left( \left( 0, 0, 0, 0, (w_j) \right); \left[ 0, \partial \zeta_{\alpha} \right], \left[ \partial \zeta_{\beta} \right], \left[ u_c \partial \zeta_{\alpha} \right], \left[ u_d \partial \zeta_{\beta} \right] \right) \in \mathcal{M}_t[\mathbf{I}];
  \]
- if \( \varepsilon_{-1} \neq 0 \) and \( \varepsilon_{-2} = 0 \), then
  \[
  \Phi_x(y) = \left( \left( 0, 0, \varepsilon_{-1}, 0, 0, (w_j) \right); \left[ \partial \zeta_{\alpha} \right], \left[ u_c \partial \zeta_{\alpha} \right], \left[ u_d \partial \zeta_{\beta} \right] \right) \in \mathcal{M}_t[\mathbf{I}(\varepsilon_{-1})];
  \]
- if \( \varepsilon_{-1} = 0 \) and \( \varepsilon_{-2} \neq 0 \), then
  \[
  \Phi_x(y) = \left( \left( 0, 0, \varepsilon_{-1}, \varepsilon_{-2}, u_c, u_d, (w_j) \right); \left[ \partial \zeta_{\alpha} \right], \left[ \partial \zeta_{\beta} \right] \right) \in \mathcal{M}_t[\mathbf{I}(\varepsilon_{-2})];
  \]
- if \( \varepsilon_{-1} \neq 0 \) and \( \varepsilon_{-2} \neq 0 \), then
  \[
  \Phi_x(y) = \left( \left( 0, 0, \varepsilon_{-1}, \varepsilon_{-2}, u_c, u_d, (w_j) \right); \left[ \partial \zeta_{\alpha} \right], \left[ \partial \zeta_{\beta} \right] \right) \in \mathcal{M}_t[\mathbf{I}(\varepsilon_{-1}, \varepsilon_{-2})].
  \]

With the expressions of \( \Phi_x \) as above, it is straightforward to check
\[
\Phi_x(0, 0, \lambda_c, \lambda_d, 0, \ldots, 0) = x,
\]
as well as to verify the statements of Lemmas 3.2, 3.4, 3.5, and Proposition 3.6 in this situation.

3.4. Universal family. Let \( \pi^w: \mathcal{C}^w \to \mathcal{M}_t \) be the universal weighted nodal curves of genus 1. The stratification \( (2.10) \) gives rise to a stratification
\[
\mathcal{C}^w = \bigsqcup_{\tau \in \mathcal{K}_R} \mathcal{C}^w_{\tau} \text{ satisfying } \pi^w(\mathcal{C}^w_{\tau}) = \mathcal{M}_t \quad \forall \tau \in \mathcal{K}_R.
\]
Parallel to \( (2.12) \) and \( (2.13) \), we set
\[
\mathcal{C}^t[\mathbf{I}] = \left( \bigotimes_{i \in \mathbf{I}} \left( \left( \Phi_i^w \left( \mathcal{C}^w \right) \right) \right) \right) \to \mathcal{C}^w \quad \forall \mathbf{I} \in \mathcal{K}^t,
\]
\[
\mathcal{C}^t = \bigsqcup_{\mathbf{I} \in \mathcal{K}^t} \mathcal{C}^t[\mathbf{I}] \to \mathcal{C}^w.
\]

Mimicking the construction of the stack structure of \( \mathcal{M}_t \) in \( (3.1) \) and \( (3.2) \), we can endow \( \mathcal{C}^t \) with a stack structure analogously. Furthermore, the projection \( \pi^t: \mathcal{C}^t \to \mathcal{M}_t \) induces a unique projection
\[
\pi^t: \mathcal{C}^t \to \mathcal{M}_t.
\]
It is straightforward that
\[
\mathcal{C}^t \cong \mathcal{C}^w \times_{\mathcal{M}_t} \mathcal{M}_t \to \mathcal{M}_t.
\]

For any scheme \( S \), a flat family \( Z/S \) of stable weighted nodal curves of genus 1 with twisted fields corresponds to a morphism \( f: S \to \mathcal{M}_t \) such that \( Z/S \) is the pullback of \( (3.31) \):
\[
Z/S \cong (S \times_{\mathcal{M}_t} \mathcal{C}^t)/S.
\]
This leads to the following statement.

**Proposition 3.9.** \( \mathcal{C}^t \to \mathcal{M}_t \) in \( (3.31) \) gives the universal family of \( \mathcal{M}_t \).
Remark 3.10. One may establish a moduli interpretation of $\tilde{M}^{\text{tf}}$ as follows: for any scheme $S$, every flat family $Z/S$ of stable weighted nodal curves of genus 1 with twisted fields can be constructed directly as follows. A priori, $Z/S$ should be over a flat family $C_S/S$ of stable weighted curves, thus by the universality of the moduli $M^{\text{wt}}$, there exists a morphism

$$\alpha : S \longrightarrow M^{\text{wt}} = \bigsqcup_{\tau \in \mathcal{F}_R^{\text{wt}}} M^{\text{wt}}_\tau$$

such that $C_S/S$ is the pullback of $C^{\text{wt}}/M^{\text{wt}}$ via $\alpha$. This induces a stratification of the scheme $S$:

$$S = \bigsqcup_{\tau \in \mathcal{F}_R^{\text{wt}}} S_\tau \quad \text{satisfying} \quad \alpha(S_\tau) \subset M^{\text{wt}}_\tau \quad \forall \tau \in \mathcal{F}_R^{\text{wt}}.$$  

(3.33)

We take

$$S^{\text{tf}}_t = \left( \prod_{\iota \in \mathbb{P}L^{\text{wt}}_t} \left( \bigoplus_{e \in \text{Edg}(i)} \alpha^*L^{\text{et}}_e \right) / S^{\text{tf}}_t \right) \pi_t^{S^{\text{tf}}} S^{\text{tf}}_t \quad \forall \left[ i \right] \in \mathcal{F}_L^{\text{wt}},$$

$$S^{\text{tf}} = \bigsqcup_{\left[ i \right] \in \mathcal{F}_L^{\text{wt}}} \pi_t^{S^{\text{tf}}} S.$$  

For any chart $\mathcal{V}_S \to S$, shrinking it if necessary, we see there exists a (smooth) chart $\mathcal{V} \to M^{\text{wt}}$ such that

$$\mathcal{V}_S \longrightarrow \mathcal{V} \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow$$

$$S \longrightarrow M^{\text{wt}}$$

commutes. The modular parameters $\zeta_e$ on $\mathcal{V}$ pull back to regular functions on $\mathcal{V}_S$, which are denoted by $\zeta^S_e$. By (3.33), we have

$$S_\tau \cap \mathcal{V}_S = \{ \zeta^S_e = 0 : e \in \text{Edg}(\tau) \} \cap \{ \zeta^S_{e'} \neq 0 : e' \notin \text{Edg}(\tau) \}.$$  

Mimicking the construction in (3.1) and (3.2), we can thus endow $S^{\text{tf}}$ with a scheme structure.

We say $Z/S$ is a flat family of stable weighted nodal curves of genus 1 with twisted fields if and only if there exists a section $\sigma^{\text{tf}}$ of $\pi_S : S^{\text{tf}} \to S$ such that

$$Z = C_S \times_S (\sigma^{\text{tf}}(S)).$$

This construction is consistent with (3.32). One can check that the groupoid sending any scheme $S$ to the set of all such defined flat families $Z/S$ is represented by $M^{\text{tf}}$.

We would like to remark that a more succinct definition of a flat family of stable weighted nodal curves of genus 1 with twisted fields should be desirable.

4. Comparison with Hu-Li's blowup stack $\tilde{M}^{\text{wt}}$

Let $\pi : \tilde{M}^{\text{wt}} \to M^{\text{wt}}$ be the sequential blowup constructed in [5] §2.2. Since $M^{\text{wt}}$ is a smooth Artin stack and the blowup centers are all smooth, so is $\tilde{M}^{\text{wt}}$. As per the convention of this paper, we omit the subscript indicating the genus. In Proposition 4.3, we show that $M^{\text{tf}}$ is isomorphic to $\tilde{M}^{\text{wt}}$. Lemma 4.2 is rather technical; it is only used in the proof of Proposition 4.3.

We briefly recall the notion of the locally tree compatible blowups described in [7] §3. Let $M$ be a smooth stack, $\gamma$ be a rooted tree, and $\mathcal{V}$ be an affine smooth chart of $M$. If there exists a set of local parameters on $\mathcal{V}$ so that a subset of which can be written as

$$\{ \zeta_e \in \Gamma(\mathcal{V}) : e \in \text{Edg}(\gamma) \},$$

then the set is called a $\gamma$-labeled subset of local parameters on $\mathcal{V}$. For example, if $M = M^{\text{wt}}$ and $\mathcal{V}$ is a chart centered at a weighted curve whose reduced dual tree is $\gamma$, then the set of the modular parameters $\{ \zeta_e \}_e \in \text{Edg}(\gamma)$ is a $\gamma$-labeled subset of local parameters.
Let $\text{Ver}(\gamma)_{\text{min}}$ be the set of the minimal vertices of $\gamma$ with respect to the tree order. We call a subset $\mathcal{S}$ of $\text{Edg}(\gamma)$ a **traverse section** if for any $v \in \text{Ver}(\gamma)_{\text{min}}$, the path between $o$ and $v$ contains exactly one element of $\mathcal{S}$. For example, the subsets $\mathcal{C}_i$ of $\text{Edg}(\gamma)$ as in (2.3) are traverse sections.

Let $\Xi(\gamma)$ be the set of the traverse sections. The tree order on $\text{Edg}(\gamma)$ induces a partial order on $\Xi(\gamma)$ such that

$$\mathcal{S} > \mathcal{S}' \iff \exists e \in \mathcal{S}, e' \in \mathcal{S}' \text{ s.t. } e \geq e'.$$

We remark that the tree order on $\text{Edg}(\gamma)$ and the induced order on $\Xi(\gamma)$ in this paper are both **opposite** to those in [7], in order to be consistent with the order of the levels of the weighted level trees.

Let $\hat{\mathcal{M}} \to \mathcal{M}$ be the sequential blowup of $\mathcal{M}$ successively along the proper transforms of the closed substacks $Z_1, Z_2, \ldots$ of $\mathcal{M}$.

**Definition 4.1.** [7, Definitions 3.2.4 & 3.2.1] The blowup $\hat{\mathcal{M}} \to \mathcal{M}$ above is said to be **locally tree-compatible** if there exists an étale cover $\{\mathcal{V}\}$ of $\mathcal{M}$ such that for each $\mathcal{V} \in \{\mathcal{V}\}$, there exist a rooted tree $\gamma$, a partition of $\Xi(\gamma)$:

$$\Xi(\gamma) = \bigsqcup_{k \geq 1} \Xi_k(\gamma)$$

and a $\gamma$-labeled subset of local parameters on $\mathcal{V}$ such that

- for every $k \geq 1$,
  $$Z_k \cap \mathcal{V} = \bigcup_{e \in \Xi_k(\gamma)} \{z_e = 0 : e \in \mathcal{S}\};$$
- if $\mathcal{S}' \in \Xi_k(\gamma)$, $\mathcal{S}'' \in \Xi_{k'}(\gamma)$, and $\mathcal{S}' > \mathcal{S}''$, then $k' < k''$.

If a sequential blowup $\hat{\mathcal{M}} \to \mathcal{M}$ is locally tree-compatible, then the blowup procedure is finite on each $\mathcal{V} \in \{\mathcal{V}\}$, because the set $\Xi(\gamma)$ is finite.

**Lemma 4.2.** If the blowup $\hat{\mathcal{M}} \to \mathcal{M}$ successively along the proper transforms of the closed substacks $Z_1, Z_2, \ldots$ of $\mathcal{M}$ is locally tree-compatible, then the blowup $\hat{\mathcal{M}}' \to \mathcal{M}$ successively along the total transforms of

$$Y_1 = Z_1, \ Y_2 = Z_1 \cup Z_2, \ Y_3 = Z_1 \cup Z_2 \cup Z_3, \ \ldots$$

yields the same space, i.e. $\hat{\mathcal{M}}' = \mathcal{M}$.

**Proof.** We prove the statement by induction. For each $h \geq 1$, we will show that after the $h$-th step, the blowup stacks $\hat{\mathcal{M}}(h)$ of $\mathcal{M}$ along the total transforms of $Y_1, \ldots, Y_h$ is the same as the blowup $\hat{\mathcal{M}}(h)$ of $\mathcal{M}$ along the proper transforms of $Z_1, \ldots, Z_h$.

The base case of the induction is trivial. Suppose the blowup $\hat{\mathcal{M}}(k) = \hat{\mathcal{M}}(k)$. We will show that for any $x \in \mathcal{M}$ and any lift $\bar{x}$ of $x$ after the $k$-th step, the blowup along the total transform $\bar{Y}_{k+1}$ of $Y_{k+1}$ has the same effect as that along the proper transform $\bar{Z}_{k+1}$ of $Z_{k+1}$ near $\bar{x}$. Since $x$ and $\bar{x}$ are arbitrary, this will establish the $(k+1)$-th step of the induction.

W.l.o.g. we may assume $x \in \bigcap_{l=1}^{k+1} Z_l$ (otherwise we simply omit the loci $Z_l$ not containing $x$ and change the indices of $Z_l$ and $Y_l$ accordingly). The blowup $\hat{\mathcal{M}} \to \mathcal{M}$ is locally tree-compatible, hence there exist a rooted tree $\gamma$, an affine smooth chart $\mathcal{V}$ containing $x$, and a $\gamma$-labeled subset of local parameters $z_e, e \in \text{Edg}(\gamma)$ on $\mathcal{V}$ such that

$$x \in \{z_e = 0 : e \in \text{Edg}(\gamma)\}.$$
Moreover, by [7, (3.13)], the total transform of each $Z_i$ with $1 \leq i \leq k$ is locally given by $\{\tilde{z}_i = 0\}$. Thus, $\tilde{Y}_{k+1}$ is locally given by $\tilde{Y}_{k+1} \cap \tilde{V}_k = (\tilde{Z}_{k+1} \cap \tilde{V}_k) \cup \{\prod_{1 \leq i \leq k} \tilde{z}_i = 0\}$.

That is, on the chart $\tilde{V}_k$, $\tilde{Z}_{k+1}$ and $\tilde{Y}_{k+1}$ are defined by the ideals

$$\mathcal{I}_{\tilde{Z}_{k+1}} = \langle z_e : e \in E(k+1) \backslash E(k), \tilde{z}_e : e \in E(k+1) \cap E(k) \rangle$$

and

$$\mathcal{I}_{\tilde{Y}_{k+1}} = \langle \tilde{z}_i \rangle_{1 \leq i \leq k},$$

respectively. Therefore, blowing up along $\tilde{Z}_{k+1}$ has the same effect on $\tilde{V}_k$ as that along $\tilde{Y}_{k+1}$. □

**Proposition 4.3.** $\mathcal{M}^{tf}/\mathcal{M}^{wt}$ is isomorphic to $\tilde{\mathcal{M}}^{wt}/\mathcal{M}^{wt}$. In particular, $\varpi : \mathcal{M}^{tf} \to \mathcal{M}^{wt}$ is proper.

**Proof.** Our goal is to construct two morphisms $\psi_1$ and $\psi_2$ between $\tilde{\mathcal{M}}^{wt}$ and $\mathcal{M}^{tf}$ so that the following diagram

$$\begin{array}{ccc}
\tilde{\mathcal{M}}^{wt} & \xrightarrow{\psi_2} & \mathcal{M}^{tf} \\
\pi & \downarrow & \downarrow \varpi \\
\mathcal{M}^{wt} & \xrightarrow{\psi_1} & \mathcal{M}^{tf}
\end{array}$$

commutes. Since $\pi$ and $\varpi$ restrict to the identity map on the preimages of the open subset

$$\{(C, w) \in \mathcal{M}^{wt} : w(C_o) > 0\} \subset \mathcal{M}^{wt},$$

respectively, we see that $\psi_2 \circ \psi_1$ and $\psi_1 \circ \psi_2$ are the identity maps. This then implies the former statement of Proposition 4.3. The latter statement follows from the former as well as the properness of the blowup $\tilde{\mathcal{M}}^{wt} \to \mathcal{M}^{wt}$.

We first construct $\psi_1$. For each $k \in \mathbb{Z}_{>0}$, let $Z_k \subset \mathcal{M}^{wt}$ be the closed locus whose general point is obtained by attaching $k$ smooth positively-weighted rational curves to the smooth 0-weighted elliptic core at pairwise distinct points. By Lemma 4.2, the blowup $\pi : \tilde{\mathcal{M}}^{wt} \to \mathcal{M}^{wt}$ successively along the proper transforms of $Z_1, Z_2, \ldots$ can be identified with the blowup of $\mathcal{M}^{wt}$ successively along the total transforms of $Y_1 = Z_1, Y_2 = Z_1 \cup Z_2, Y_3 = Z_1 \cup Z_2 \cup Z_3, \ldots$

We observe that for each $k \in \mathbb{Z}_{>0}$, the pullback $\varpi^{-1}(Y_k)$ to $\mathcal{M}^{tf}$ is a Cartier divisor. In fact, for any $[\ell] = [\gamma, \mathcal{W}, \ell] \in \mathcal{L}^{wt}$ and $x \in \mathcal{M}^{tf}$, let $\mathcal{U} \to \mathcal{M}^{tf}$ be a twisted chart centered at $x$, lying over a chart $\mathcal{V} \to \mathcal{M}^{wt}$. In [5, the blowup $\pi$ locally on $\mathcal{V}$ is proved to be compatible with the weighted tree $(\gamma, \mathcal{W})$ obtained by contracting all the edges $e$ of $\gamma$ as long as there exists $\nu \geq \nu_e$ satisfying $w(\nu) > 0$. Let $\{z_e : e \in E(\gamma)\}$ be a set of modular parameters on $\mathcal{V}$ as in (3.1) and

$$\{z_e\}_{e \in E(\gamma)} \cup \{u_e\}_{e \in E(\gamma) \backslash \{e_0\} \cap E(\gamma)} \cup \{\tilde{z}_e\}_{e \in E(\gamma)}$$

be the subset of the parameters (3.9) on $\mathcal{U}$. We claim that

$$\varpi^{-1}(Y_k) \cap \mathcal{U} = \{\prod_{i \in I_+} \tilde{z}_i = 0\}.$$

To show (4.1), we first notice that $\varpi^{-1}(Y_k) \cap \mathcal{U} = \varpi^{-1}(Y_k \cap \mathcal{V})$ by Corollary 3.7. Every irreducible component of $Y_k \cap \mathcal{V}$ can be written in the form

$$Y_k, \mathcal{G} := \{z_e = 0 : e \in \mathcal{G}\}$$

with $\mathcal{G} \in \Xi(\gamma)$, $|\mathcal{G}| \leq k$, $\mathcal{G} \cap (E(\gamma) \backslash \mathcal{U}_m) \neq \emptyset$. 

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For each irreducible component $Y_{k, \mathcal{S}}$ of $Y_k \cap \mathcal{V}$, the local expression $\theta_x$ of $\varpi$ as in (3.12) implies that the pullback $\varpi^{-1}(Y_{k, \mathcal{S}})$ can be written as
\begin{equation}
\left\{ \prod_{h \in [\ell(e), \ell_v(\varepsilon_h)]} \varepsilon_h = 0 : e \in \mathcal{S} \cap (\widetilde{\text{Edg}}(t) \backslash \mathbb{I}_m) \right\}.
\end{equation}

Therefore, $\varpi^{-1}(Y_{k, \mathcal{S}}) \cap \mathbb{I} = \{ \varepsilon_i = 0 \} \quad \forall i \in \mathbb{I}_+$ with $|\mathcal{E}_i| \leq k$.

Since every $\varpi^{-1}(Y_k)$ is a Cartier divisor of $\mathcal{M}^\text{tf}$, by the universality of the blowup $\pi : \widetilde{\mathcal{M}}^\text{wt} \to \mathcal{M}^\text{wt}$, we obtain a unique morphism $\psi_1 : \mathcal{M}^\text{tf} \to \widetilde{\mathcal{M}}^\text{wt}$ that $\varpi : \mathcal{M}^\text{tf} \to \mathcal{M}^\text{wt}$ factors through.

We next construct $\psi_2$ explicitly. For any $\widetilde{x} \in \widetilde{\mathcal{M}}^\text{wt}$, let $(C, w)$ be its image in $\mathcal{M}^\text{wt}$. As shown in [7, §3.3], there exists a unique maximal system of exceptional divisors $\widetilde{E}_{i_1}, \ldots, \widetilde{E}_{i_k} \subset \widetilde{\mathcal{M}}^\text{wt}, \quad 1 \leq i_1 < \cdots < i_k$

containing $\widetilde{x}$. Each $\widetilde{E}_{i_j}$ is obtained from blowing up along the proper transform of $Z_{i_j}$. Note that $k$ is possibly 0, which means $(C, w)$ is not in the blowup loci. The weighted dual tree $\tau = (\gamma, C, w)$, along with the exceptional divisors $\widetilde{E}_{i_1}, \ldots, \widetilde{E}_{i_k}$, uniquely determines a weighted level tree $t_x$ such that
\[ \mathbb{I}_+ = \mathbb{I}_+ (t_x) = \{-i_k, \ldots, -i_1\} \].

In particular, $m = m(t_x) = -i_k$.

With the line bundles $L_{e, e} \in \text{Edg}(t_x)$, as in (2.11), the notation $[\cdot, \cdot]_{t_x}$ and $[\cdot, \cdot]_{t_x}$ as in (2.2), and the notation $i[h]$ as in (3.6), the line bundles
\begin{equation}
\mathcal{L}_i = L_{e, e} \otimes \bigotimes_{\ell(e), [\ell(e)]_{t_x}} \mathcal{L}_{i} \quad \to \quad \mathcal{M}^\text{wt}_{\tau}, \quad i \in \mathbb{I}_+,
\end{equation}
can be constructed inductively over $\mathbb{I}_+$. Then, we take
\begin{equation}
\mathcal{L}_e = L_{e, e} \otimes \bigotimes_{\ell(e), [\ell(e)]_{t_x}} \mathcal{L}_{i} \quad \to \quad \mathcal{M}^\text{wt}_{\tau}, \quad e \in \text{Edg}(t_x).
\end{equation}

In particular, $\mathcal{L}_i = \mathcal{L}_i$. For each $e \in \text{Edg}(t_x)$, (4.3) and (4.4) imply
\[ \mathcal{L}_{i} \otimes \bigotimes_{e' > e} (\mathcal{L}_{e'} \otimes \mathcal{L}_{i(e')}) = L_{e} \otimes \bigotimes_{e' > e} (\mathcal{L}_{i(e')}) \].

Hence for each $i \in \mathbb{I}_+$,
\begin{equation}
\mathcal{P} \left( \bigoplus_{\ell(e) = i} (\mathcal{L}_{e} \otimes \bigotimes_{e' > e} (\mathcal{L}_{e'} \otimes \mathcal{L}_{i(e')})) \right) = \mathcal{P} \left( \bigoplus_{\ell(v_e) = i} L_{e} \right) .
\end{equation}

For $h \geq 1$, let $\widetilde{x}_{(h)}$ be the image of $\widetilde{x}$ in the exceptional divisor of the $h$-th step. Given $i \in \mathbb{I}_+$, the proper transform of $Z_{-1}$ after the first $-i-1$ steps of the blowup may have several connected components; see [7, Lemma 3.3.2]. The normal bundle of the component containing $\widetilde{x}_{(-i-1)}$ is the pullback $\pi_{(-i-1)}^* \bigoplus_{e \in \mathbb{I}_-} \mathcal{L}_e$, where $\pi_{(h)} : \widetilde{\mathcal{M}}^\text{wt}_{(h)} \to \mathcal{M}^\text{wt}$ is the blowup after the $h$-th step.
Notice that the non-zero entries of $x_{i-j}$ exactly correspond to the edges $e \in \mathcal{E}_i$ satisfying $\ell(v_e^\circ) = i$. Therefore,
\[ x_{i-j} \in \pi_{i-j}^\circ \left( \bigoplus_{\ell(v_e^\circ) = i} \mathcal{L}_e \right). \]

Then, $x_{i-j}$ with $j \in [i, 0]_{t_2}$ together determine a unique
\[ \eta_{i,j}(x) \in \hat{\mathcal{P}} \left( \bigoplus_{\ell(v_e^\circ) = i} \left( \mathcal{L}_e \otimes \mathcal{L}_e^{\sigma} \right) \right) = \hat{\mathcal{P}} \left( \bigoplus_{\ell(v_e^\circ) = i} L_e^\circ \right). \]

The last equality above follows from (4.5). We then set
\[ \psi_2(x) = \left( (C, w), [t_2], (\eta_{i,j}(x)) : i \in \mathbb{H}_+(t_2) \right) \in \mathcal{M}_h^I. \]

Obviously, this implies $\tau \circ \psi_2 = \pi$.

It remains to verify such defined $\psi_2$ is a morphism. Let $\mathcal{V} \to \mathcal{M}^I$ be a smooth chart containing $(C, w)$, and $\{ \zeta : e \in \text{Edg}(t_2) \} \cup \{ \zeta_j : j \in J \}$ be a set of local parameters centered at $(C, w)$ as in (3.3). As shown in [7, §3.1&§3.3], there exists a chart $\tilde{\mathcal{V}} \to \mathcal{M}^I$ containing $x$ with local parameters
\[ \tilde{\mathcal{V}} = (\mathcal{V} \setminus (\mathbb{I}_m \cup \{ e_i : i \in \mathbb{I}_+ \})) \]

All $\rho_e$ are nowhere vanishing on $\tilde{\mathcal{V}}$. Moreover, with $\pi : \tilde{\mathcal{V}} \to \mathcal{V}$ denoting the blowup, we have
\[ \pi^* \zeta_i = \prod_{e \in \text{Edg}(t_2) \setminus \mathbb{I}_m} \prod_{i} \tilde{\zeta}_i \quad \forall e \in \mathbb{I}_+; \quad \pi^* \zeta_e = \rho_e \prod_{e \in \text{Edg}(t_2) \setminus \mathbb{I}_m} \tilde{\zeta}_i \quad \forall e \in \mathbb{I}_-. \]

For $e \in \{ e_i : i \in \mathbb{I}_+ \}$, we set $\rho_e = 1$. Then,
\[ \rho_e \in \Gamma \left( \mathcal{O}^\circ_{\mathcal{V}_2} \right) \quad \forall e \in \text{Edg}(t_2) \setminus \mathbb{I}_m. \]

Let $\mathcal{U}_{\psi_2(x)} \to \mathcal{M}^I$ be a twisted chart centered at $\psi_2(x)$, lying over $\mathcal{V} \to \mathcal{M}^I$. The parameters on $\mathcal{U}_{\psi_2(x)}$ are as in (3.3). It is a direct check that the point-wise defined $\psi_2$ can locally be written as
\[ \psi_2 : \tilde{\mathcal{V}}_2 \to \mathcal{U}_{\psi_2(x)} \]

such that
\[ \psi_2^* \tilde{\zeta}_i = \tilde{\zeta}_i \quad \forall i \in \mathbb{I}_+; \quad \psi_2^* u_e = \prod_{e \geq e_i} \rho_e \prod_{e \leq e_i} \rho_e \quad \forall e \in \text{Edg}(t_2) \setminus (\mathbb{I}_m \cup \{ e_i : i \in \mathbb{I}_+ \}); \]
\[ \psi_2^* z_e = \zeta_e \quad \forall e \in \mathbb{I}_m; \quad \psi_2^* z_j = \zeta_j \quad \forall j \in J. \]

This shows $\psi_2 : \mathcal{M}^I \to \mathcal{M}^I$ is a morphism. 

\begin{remark}
In [6], another resolution $\mathcal{M}^\text{dr} \to \mathcal{M}^I$, called the derived resolution of $\mathcal{M}^I$, is constructed for the purpose of diagonalizing certain direct image sheaves. That resolution is “smaller” in that the resolution $\mathcal{M}^\text{dr} \to \mathcal{M}^I$ of [5] factors through $\mathcal{M}^\text{dr} \to \mathcal{M}^I$. Mimicking the approach of §3, we may construct a moduli stack
\[ \mathcal{M} = \bigsqcup_{[t_2]} \mathcal{M}_{h_{[t_2]}}, \quad \mathcal{M}_{h_{[t_2]}} = \hat{\mathcal{P}} \left( \bigoplus_{e \in \text{Edg}(t_2), \ell(v_e^\circ) = m(t)} L_e^\circ \right) \to \mathcal{M}^I_{h_{[t_2]}}. \]

This moduli should be isomorphic to $\mathcal{M}^\text{dr}$.
\end{remark}
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