Towards a smooth compactification of the space of curves in $\mathbb{P}^n$ (based on joint works with Yi Hu and Jun Li)

**Object:** $\overline{M}_g(\mathbb{P}^n, d) = \{ \gamma : C \to \mathbb{P}^n \mid C \text{ nodal, } g(C) = g, \deg \gamma = d, |\text{Aut}(\gamma)| < \infty \}$

**Pros:**
- A compact space containing $\mathcal{M}_g(\mathbb{P}^n, d) = \{ \gamma : C \to \mathbb{P}^n \mid C \text{ smooth, } g(C) = g, \deg \gamma = d \}$
- Carries a virtual fundamental class $\rightsquigarrow$ Gromov-Witten invariants

**Cons:**
- Boundary is too large: $\mathcal{M}_g(\mathbb{P}^n, d)$ is usually not dense in $\overline{M}_g(\mathbb{P}^n, d)$.
- How to describe the closure (i.e., the main component)?
- Can be arbitrarily singular (when $g$ and $d$ vary).
- How to resolve the singularities?

**Q1:** Can we construct $\overline{M}_g(\mathbb{P}^n, d) \to \overline{M}_g(\mathbb{P}^n, d)$ s.t.

1. $\overline{M}_g(\mathbb{P}^n, d)$ has smooth irreducible components and at worst normal crossing singularities (i.e., $\varnothing < \odot$)

2. Proper
3. birationally dominating the main component?

**Remark.** An affirmative answer could lead to a direct approach to GW of CY 3-folds $\rightleftarrows$ mirror symmetry and shed some light on the resolution of other singular spaces.

**Answer:**

$g = 1$: Yes. (Vakil, Zinger '08; Hu, Li '10; Ranganathan, Santos-Paulo, Wise '17; Hu, '19)

$g = 2$: Yes. (Hu, Li '18; Hu, '20; Battistella, Carocci '20)

$g \geq 3$: ?
Where are the singularities?

- \( U : C \to \mathbb{P}^n \) can be described by \( (C, s_0, \ldots, s_n) \) with
  \[
  s_i = u^* x_i \in H^0(C, u^* \mathcal{O}_{\mathbb{P}^n}(1)).
  \]
  \( (C, u) \) is singular whenever \( h^0(C, u^* \mathcal{O}_{\mathbb{P}^n}(1)) \) jumps.

- Let \( \mathcal{L} \xrightarrow{f} \mathbb{P}^n \) \( (C, u, x) \xrightarrow{\phi} U(x) \) be the universal family.

  \[
  \xymatrix{ & \mathcal{L} \ar@{-->}[d]^f \ar[d] \ar@{-->}[r] & \mathbb{P}^n \ar[d] & \ar[d] \ar@{-->}[r] & U(x) \ar[d] \ar[r] & (C, u) \ar[d]}
  \]

  Then \( (C, u) \) is singular \( \iff \) the sheaf \( R^0 \pi_* f^* \mathcal{O}_p(x) \) is not locally free.

- The local structure of \( \overline{M}_g(\mathbb{P}^n, d) \) is encoded in the derived object
  \( R^0 \pi_* f^* \mathcal{O}_p(x) \).

- Fix \( (C, u) \in \overline{M}_g(\mathbb{P}^n, d) \) and a small open neighborhood \( U \) of \( (C, u) \).

  To study \( U \), we hope to
  - embed \( U \) in something smooth \( \text{(Ev later on)} \) and
  - write the local equation for \( U \) in \( Ev \)

- How to construct such \( Ev \)?
  
  Let \( D_g = \{ (C, D) \mid C \text{ normal, } g(C) = g, \text{ D effective, stability} \} \)
  \( D_g \) is a smooth algebraic stack.

  \[
  U \to D_g, \quad (C, u = [u_0 : \ldots : u_n]) \to (C, u^* \mathcal{O}(1)) \]
  (\( \leq \overline{M}_g(\mathbb{P}^n, d) \))

  May assume simple

  Let \( U \to D_g \) be a smooth chart containing the image of \( U \).

  \( C = \{ (c, D, x) \mid (c, D) \in U, x \in C \} \)

  \( D = \{ (c, D, D) \mid (c, D) \in U \} \)

  \( \xymatrix{ C \ar@{-->}[r] & D } \)

  universal curve

  the universal divisor on \( C \).
We can choose $g$ sections $A_1, \ldots, A_g$ of $\mathcal{F} : \mathcal{E} \to \mathcal{V}$ s.t.
$$R^i p_\ast \mathcal{O}_\mathcal{E}(D + A) = 0, \quad R^i p_\ast \mathcal{O}_\mathcal{E}(D + A) \text{ locally free}$$
:= A_1, \ldots, A_g$

Let $E_V$ : the total space of $p_\ast \mathcal{O}_\mathcal{E}(D + A)^\otimes_n$ \(\text{dim of } \mathbb{P}^n\)

\[\mathcal{F} : p_\ast \mathcal{O}_\mathcal{E}(D + A)^\otimes_n \to p_\ast \mathcal{O}_\mathcal{E}(D + A)^\otimes_n|_A : \text{tautological restriction}\]

\[\text{trivial vector bundles over } \mathcal{V}\]

**Theorem** : (Hu, Li '10):

There is a canonical open immersion $U \to \text{ker} \mathcal{F}$.

- From now on, we will put $U < \overline{M}_g(\mathbb{P}^n, d)$ aside and focus on $\text{ker} \mathcal{F}$:

  \[\mathcal{M}_g. \quad \bar{w} = 0\]

  \(< \text{local equation for } \overline{M}_g(\mathbb{P}^n, d)\>

  where $\mathcal{M}_g$ is a $g \times d$ matrix and $\bar{w}$ is a $d \times n$ matrix of free variables.

- To use the local equations and obtain a resolution

  \[\tilde{M}_g(\mathbb{P}^n, d) \to \overline{M}_g(\mathbb{P}^n, d)\]

  one needs

  (I) to write $M_g$ as explicit as possible (this should suffice to get local resolution)

  (II) to globalize (I) (nice for $g = 1$; complicated for $g = 2$).

\[\exists \ x \ 1\]

$g = 1$

\[\text{map constant} \quad \text{map not constant} \quad \rightarrow \quad \text{map constant}\]
On the chart $U \to D_1$, local parameters $t_1, \ldots, t_d$

$\mathcal{B}_a = 0 \implies$ node $a$ not smooth.

$M_p = [t_1, t_2, t_3, t_4, 0, \ldots]$

$M_p \cdot \mathbf{w} = 0 \implies s_a \omega^1 + s_3 \omega^2 + s_4 \omega^3 = 0, 1 \leq j \leq n \implies \mathbf{w}$ singular.

Locally, how to desingularize $D_1$?

Blowing up along $s_a = s_3 = 0$.

The local blowup is embedded in $U \times \mathbb{P}^1$, covered by two charts:

On the chart $U \times \{u_1 = 1\}$,

$M_p$ pulls back to $\mathcal{B}_a [1, u_b t_c, u_b t_d, 0, \ldots]$

$\implies 0$ pulls back to $s_a (\omega^1 + u_b t_c \omega^2 + u_b t_d \omega^3) = 0, 1 \leq j \leq n

smooth, smooth, main

normal crossing

On the chart $U \times \{u_1 = 1\}$,

$M_p$ pulls back to $\mathcal{B}_b [u_a, t_c, t_d, 0, \ldots]$

$\implies 0$ pulls back to $s_b (u_a \omega^1 + s_c \omega^2 + s_d \omega^3) = 0, 1 \leq j \leq n \implies \mathbf{w}$ not smooth yet along $u_a = s_c = s_d = 0$

So we continue to blow up along $u_a = s_c = s_d = 0$. 
The local blowup is embedded in \( (V \times \{ u_a = 1 \}) \times \mathbb{P}^2 \), covered by \( \mathcal{V}_{\mathfrak{a}} : \mathfrak{c} \cdot y : \mathfrak{d} \).

chests \( V \times \{ u_a = 1 \} \times \{ u_c = 1 \} \), on which \( \varphi \) pulls back to \( M_p = \mathfrak{p}_a u_a [ \mathfrak{d}, u_c, u_d, \ldots ] \). \( \mathfrak{c} \cdot u_e : (w^d + y_c w^c + y_d w^d) = 0 \)

smooth bdries \( \text{smooth main} \)

normal crossing

\( V \times \{ u_a = 1 \} \times \{ u_c = 1 \}, \text{ similar} \)

\( V \times \{ u_a = 1 \} \times \{ u_d = 1 \}, \text{ similar} \)

\( V \times \{ u_a = 1 \} \times \{ u_c = 1 \}, \text{ similar} \)

- These local blowups are global and have clear geometric descriptions.

blowing up \( \mathbb{D}^2 \) along \( \{ \circ \} \), then \( \{ \circ \} \), then \( \{ \circ \} \), ...

**Theorem** (Ueki, Zinger '08, Hu, Li '10, Raghavan, Santos-Parker, Wise '17 (log geom.))

The above blowups give an affirmative answer to \( \mathbb{Q} \) for genus \( 1 \).

**Remark.** The local equations \( M_p \cdot \mathfrak{w} = 0 \) also lead to a description of \( (c, \mathfrak{u}) \in \text{Closure} (M, (\mathbb{P}^n, d)) \setminus M, (\mathbb{P}^n, d) \)

( Battistella, Carocci, Manolache '18; originally by Zinger via symplectic topology.)

- Next, consider \( g \geq 2 \).

Recall: The general idea is still to

1. describe \( M_p \) explicitly and know how to desingularize locally

2. globalize (1).

- (2) is fine.

**Theorem** (Hu, Li, '18)
Theorem (Hu, Li, 18)

\[
M_\gamma = \begin{pmatrix}
\cdots & C_{i \iota} & T & S_e & & C_{i j} & T & S_e & \cdots \\
\text{nodes } g_e \text{ between } & S_i & \text{ and } C_i & & & & S_j & \text{ and } C_j & \\
\cdots & C_{i \eta} & T & S_e & & C_{j \eta} & T & S_e & \cdots \\
\text{nodes } g_e \text{ between } & S_i & \text{ and } C_2 & & & & S_j & \text{ and } C_2 & \\
\end{pmatrix}
\]

where \( C_{i \iota} \) are nowhere vanishing functions on \( U \), satisfying

\[
\det \begin{pmatrix} C_{i \iota} & C_{i j} \\ C_{j \iota} & C_{j j} \end{pmatrix} = k_{ij} \cdot \prod \begin{pmatrix} S_e \\ \text{common nodes } g_e \text{ between } S_i, S_j \text{ and the smallest genus 2 subcurve} \end{pmatrix}
\]

a local function measuring how far \( S_j \) is from being conjugate to \( S_i \) and \( S_j \) are

\( \mathbb{D} \) is complicated.

(\( \mathbb{D} \) is complicated.

\[
\begin{array}{r}
E, 2, \quad g = 2 \\
(\mathbb{C}, U) \in M_2(\mathbb{P}^n, \mathbb{D}) \\
(\mathbb{C}, D = S_i \cup \cdots \cup S_j) \in \mathbb{D}_2
\end{array}
\]

(assume \( a, b \) in general position, i.e.
not Weierstrass/Conjugate)

Under suitable trivialization,

\[
M_\gamma = \begin{bmatrix}
S_a & S_a & S_b & \cdots \\
0 & S_a & S_b & \cdots \\
0 & S_a & S_b & \cdots \\
\end{bmatrix} \quad \cdots (3)
\]

\[
= \begin{bmatrix}
S_b & S_b & S_a & \cdots \\
0 & S_b & S_a & \cdots \\
0 & S_b & S_a & \cdots \\
\end{bmatrix} \quad \cdots (4)
\]

\( \mathbb{H}_L, \mathbb{H}_L \ldots \mathbb{H}_L \cdots \mapsto \mathbb{R} - \mathbb{Q} - \mathbb{Q} \leftrightarrow \text{globally along } \{ \mathbb{C} \leftrightarrow \} \)
Locally, blow up $S_4 = S_6 = 0$ $\leftrightarrow$ globally, along $\{\phantom{\circ} \}$.

The local blowup is embedded in $V \times P^1$, covered by two charts $[u_\alpha : u_\beta]$.

On the chart $V \times \{u_\alpha = 1\}$, (3) pulls back to

$$
\begin{bmatrix}
1 & * & * & \ldots \\
0 & S_4 & U_6 & \ldots \\
\end{bmatrix}
$$

$g_2 = 0$

Exceptional divisor

The 2nd row still does not give smooth equation.

$\Rightarrow$ Locally, need to blow up $S_4 = U_6 = 0$ $\leftrightarrow$ the locus $\{1, x\}$ in the exceptional divisor.

$\Rightarrow$ Globally, along $\mathbb{P}(L_\alpha \oplus L_\beta) = \mathbb{P}(L_\alpha \oplus L_\beta)$.

On the chart $V \times \{u_\alpha = 1\}$, use (4) instead of (3).

Remark. Ex. 2 suggests

- only blowing up along $\{\phantom{\circ} \}$, $\{\phantom{\circ} \}$, $\ldots$ is not enough.

needs more rounds of blowups.

- The blowup centers of some rounds lie in the exceptional divisors of previous rounds.

$\bigcirc \ \text{(Ex. 3)} \ \ g = 2$

$\Rightarrow \rightarrow \ \ \begin{array}{c}
\bigcirc \\
\oplus
\end{array}$

assume $\deg D \mid_{R_2} \geq 3$
Under suitable trivialization
\[
M_q = \begin{bmatrix}
1 & * & * & \ldots & s_a & \cdots \\
0 & k_{i2} & k_{i3} & \cdots & s_a & \cdots \\
R_1 & & & & & \\
0 & s_1 & s_2 & \cdots & s_a & \cdots \\
R_2 & & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
\end{bmatrix}
\]

Locally, blow up along
\[s_p = s_q = s_a = 0 \iff \text{globally, along } \{ \} \]

The blowup is embedded in \( V \times \mathbb{P}^3 \)
\( [u_p : v_q : u_a] \)

On the chart \( V \times \{ u_a = 1 \} \), similar to Ex 1 \& Ex 2.

On the chart \( V \times \{ u_p = 1 \} \), \( M_q \) pulls back to
\[
\begin{bmatrix}
1 & 0 \\
0 & s_p \\
\end{bmatrix}
\begin{bmatrix}
1 & * & * & \ldots & * & \ldots \\
0 & s_1 & s_2 & \cdots & s_a & \cdots \\
\end{bmatrix}
\]

at least one of them is nowhere vanishing @ (c,\( \overrightarrow{D} \)) (hence on \( V \) ) bloc

On the chart \( V \times \{ u_q = 1 \} \), similar.

Remark. Ex 3 suggests a round of blowups: \( \{ \} \), then \( \{ \} \), \( \{ \} \), \( \ldots \)
which is different from Ex 2.

So the order of these rounds of blowups needs attention.

Theorem (Hu, Li, - '18):
For genus \( = 2 \), a \( g \)-round blowup gives an affirmative answer of \( Q_1 \).
Remark. The first few rounds are along:

\[ \{ \text{ } \} \to \{ \text{ } \} \to \{ \text{ } \} \to \cdots \]

\[ \{ \text{ } \} \to \{ \text{ } \} \to \{ \text{ } \} \cup \{ \text{ } \} \to \cdots \]

\[ \{ \text{ } \} \to \{ \text{ } \} \to \{ \text{ } \} \to \cdots \]

\[ \{ \text{ } \} \to \{ \text{ } \} \to \{ \text{ } \} \to \cdots \]

\[ \cdots \]

- The exceptional divisors are accumulating. We need a better way to go from local equations to global blowups. To this end, we introduce the theory of "stacks with twisted fields" to reorganize the $g=1, 2$ cases and hopefully to treat higher genera.

- To get some idea, let's revisit Ex. 1. from another prospective:

**Ex. 4. (Ex. 1 revisited):**

\[ (C, u) \in \bar{M}_1 \left( \mathbb{P}^h, d \right) \]

\[ M_y = [ y_1, \ldots, y_n, z_1, \ldots, z_d, 0, \ldots ] \]

- Notice that each $(C, D) \in D_1$ corresponds to a unique rooted tree $\mathcal{T}(C, D)$ as follows:
This gives a stratification: \( \mathcal{D}_1 = \bigsqcup \mathcal{D}_c \)

\( \mathcal{D}_c \cap V = \{ S_a = S_b = S_c = S_d = 0 \} \), they give rise to line bundles \( L_a, L_b, L_c, L_d \)

In order to desingularize \( \text{ker} \ M_\mu \), we need to compare how fast
\[ S_a, S_b S_c, S_d S_d \rightarrow 0 \]

by adding levels to \( \Lambda \):

\[ \ldots \]

For each leveled tree, we add
\[ P(L_a), P(L_a \otimes L_b), \ldots, \]
\[ P(L_a \otimes L_b \otimes L_c \otimes L_d) \]

\( \mathcal{D}_\mu \)

a bunch of strata.

For all rooted trees and all level structures on them, we obtain a number of strata.

**Theorem.** (Hu, '19)

These strata can be glued together using smooth charts in a canonical way to form a smooth \( \mathcal{D}_1^{tf} \rightarrow \mathcal{D}_1 \), birationally and properly dominating \( \mathcal{D}_1 \).

This construction desingularizes \( \overline{M}_1(4^b, d) \).

**Remark.** This construction is actually isomorphic to the Vakil-Zinger/Hu-Li blowup construction.
The advantage (in the sense of resolution of singularities) is we do not need to describe the blowup centers.

This theory does not only work for $\mathcal{D}_1$. For any smooth $M$, if it has a stratification $M = \bigsqcup_{a \in I} M_a$ by smooth subspaces (substacks) satisfying

1. Each $M_a$ corresponds to a unique tree $T_a$,
2. Each edge corresponds to a line bundle $L_e \to M_a$ (locally gives a normal direction of $M_a$ in $M$),
3. Some compatibility conditions,

Then $M^{tf} \to M$ can be constructed by mimicking $D_1^{tf} \to D_1$.

Moreover, just as $D_1^{tf}$, the new $M^{tf}$ has a natural stratification. Often, there are choices of assigning rooted trees to the strata of $M^{tf}$, satisfying 1) - 3) above. This suggests a recursive construction:

$$\ldots \to (M^{tf})^{tf} \to M^{tf} \to M$$

**Theorem.** (Hu, ’20)

We can apply this construction 8 times, starting from $\mathcal{D}_2$, to obtain an affirmative answer for $\mathcal{D}_4$ when genus = 2.

Thank you!