• The purpose of this note is to classify groups of order 2015. This was a qualifying exam problem, so consider carefully whether or not you want to read a solution of it.

• Let $G$ be a group of order $2015 = 5 \cdot 13 \cdot 31$. By the first Sylow theorem, $G$ has Sylow $p$-groups for each prime dividing 2015. Let $P, Q, R$ be the respective 5, 13, 31-Sylow subgroups.

• Since $n_{31}$ divides $5 \cdot 13$ and is congruent to 1 mod 31, $n_{31} = 1$. Therefore, $R$ is a normal subgroup of $G$.

• Since $n_{13}$ divides $5 \cdot 31$ and is congruent to 1 mod 13, $n_{13} = 1$. Therefore, $Q$ is a normal subgroup of $G$.

• Since $n_5$ divides $13 \cdot 31$ and is congruent to 1 mod 5, $n_5 = 1$ or $n_5 = 31$.

• If $n_5 = 1$, then $P$ is also a normal subgroup of $G$. Since 5, 13, 31 are coprime, Lagrange’s theorem implies that the pairwise intersections of $P, Q, R$ are trivial. Therefore $G$ is the direct product of $P, Q, R$. Since they are cyclic of coprime orders, $G$ is the cyclic group of order 2015.

• Suppose $n_5 = 31$ and that $G$ is non-abelian. Since $Q$ and $R$ are normal subgroups of $G$, $QR$ is a normal subgroup of $G$. Since $Q \cap R = 1$, $QR$ is the (internal) direct product of $Q$ and $R$. Since $Q$ and $R$ are cyclic of coprime order, $QR$ is a cyclic group of order $13 \cdot 31 = 403$. Furthermore, $QR$ is a normal subgroup of $G$, $P \cap QR = 1$ (again, by Lagrange’s theorem), and $G = PQR$ (since $QR$ has index 5 and the elements of $P$ belong to different cosets of $G/QR$), we know that $G$ is a semi-direct product of $P$ and $QR$. So $G \cong QR \rtimes \phi P$ for some group homomorphism $\phi : P \to \text{Aut}(QR)$. Since $QR$ is cyclic of order 403, $\text{Aut}(QR)$ is cyclic of order $\phi(403) = (13 - 1)(31 - 1) = 360$. (The relationship is that $\text{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$ acts on $\mathbb{Z}/n\mathbb{Z}$ as multiplication.)

• Let $x$ be a generator of $P$ and let $y$ be a generator of $QR$. Because $G$ is non-abelian, $\psi$ is not the trivial map. Since $P$ is simple, this implies that $\psi$ is injective. Therefore, $\psi(x) = \sigma$ is an element of $\text{Aut}(QR)$ of order 5, and we have $xyx^{-1} = \sigma(y)$.

• I claim that the choices of $x, y,$ and $\sigma$ do not affect the structure of the group. If $\tau \in \text{Aut}(QR)$ is another element of order 5, then $\tau = \sigma^n$ for some $n$ (coprime to 5) because a cyclic group of order 360 has a unique (cyclic) subgroup of order 5. Therefore, $x^n y x^{-n} = \sigma^n(y) = \tau(y)$. Since $P$ has order 5 and $n$ is coprime to 5, the $n$th power map $P \to P$ is an automorphism. Therefore, changing which element $\psi(x)$ is equivalent to picking another generator of $P$. Since $y$ is an arbitrary generator of $QR$, the relation $xyx^{-1} = \sigma(y)$ holds for all of them.

• Now we need only give an example of a specific choice of $\sigma$ to make this explicit. By the Chinese remainder theorem, $(\mathbb{Z}/403\mathbb{Z})^\times \cong (\mathbb{Z}/13\mathbb{Z})^\times \times (\mathbb{Z}/31\mathbb{Z})^\times$. We want an element of order 5. The relation $n^5 \equiv 1 \pmod{13}$ implies that $n \equiv 1 \pmod{13}$ because 5 does not divide 12 = $\phi(13)$ (this is another application of Lagrange’s theorem). On the other hand, 2 has multiplicative order 5 modulo 31. So we want a number $n$ such that $n \equiv 1 \pmod{13}$ and $n \equiv 2 \pmod{31}$. The Euclidean algorithm shows us that

$$1 = 12 \cdot 13 - 5 \cdot 31.$$ 

So we want a number equivalent to $2 \cdot (12 \cdot 13) + 1 \cdot (-5 \cdot 31) \equiv 403 \pmod{403}$, and in fact 157 does the trick. Thus a presentation for our group is

$$\langle x, y : x^5 = y^{403} = 1, xyx^{-1} = y^{157} \rangle.$$