The purpose of this note is to show that taking fixed points commutes with base change when working with $G$-modules. We make no assumptions on the cardinality of $G$ nor the characteristic of the base ring. Additionally, the identification is canonical.

Let $G$ be a group with subgroup $H$, and let $M$ be a $A[G]$ module for some commutative ring, $A$, with unit. Furthermore, let $B$ be a flat, commutative, unital, $A$-algebra. As a specific case, $B/A$ can be a field extension and $M$ can be a vector space over $A$ on which $G$ acts linearly.

Let $\bigoplus_{h \in H} M$ be a $G$-module, given by the action

$$g \cdot (x_h)_h = (y_h)_h,$$

where $y_h = g x g^{-1} h g$.

We show that this is an action as follows. Let $g_1, g_2 \in G$ and $(x_h)_h \in \bigoplus_{h \in H} M$. Let

$$g_1 \cdot (x_h)_h = (y_h)_h \text{ and } g_2 \cdot (y_h)_h = (z_h)_h.$$

Then we have

$$y_h = g_1 x g_1^{-1} h g_1 \text{ and } z_h = g_2 y g_2^{-1} h g_2 = g_2 \left( g_1 x g_1^{-1} (g_2^{-1} h g_2) h_1 \right) = (g_2 g_1) x (g_2 g_1)^{-1} h (g_2 g_1).$$

Therefore, $(g_2 g_1) \cdot (x_h)_h = (z_h)_h$, and this is indeed an action.

Define the map

$$f_M : M \to \bigoplus_{h \in H} M : m \mapsto (m - hm)_h \in H.$$

Let $g \in G$ and note that

$$f_M(gm) = (gm - hgm)_h = (g(m - (g^{-1} h g)m))_h = g \cdot f_M(m).$$

It is clear that $f_M$ is $A$-linear, so it is a homomorphism of $A[G]$-modules.

Note that $M^H$ is the kernel of $f_M$. So we have the following exact sequence of $A[G]$-modules:

$$0 \to M^H \to M \overset{f_M}{\to} \bigoplus_{h \in H} M.$$

Since $B$ is flat, $- \otimes_A B$ is an exact functor. Therefore,

$$0 \to M^H \otimes_A B \to M \otimes_A B \overset{f_M \otimes \text{id}}{\to} \bigoplus_{h \in H} M \otimes_A B$$

is an exact sequence of $B[G]$-modules. Since tensor products distribute over direct sums,

$$\left( \bigoplus_{h \in H} M \right) \otimes_A B \cong \bigoplus_{h \in H} (M \otimes_A B)$$

as $B$-modules. It is easy to see that this homomorphism is $G$-equivariant. Thus we have an exact sequence of $B[G]$-modules,

$$0 \to M^H \otimes_A B \to M \otimes_A B \overset{f_M \otimes \text{id}}{\to} \bigoplus_{h \in H} (M \otimes_A B).$$

Since $f_M \otimes \text{id} = f_M \otimes_A B$, we canonically have

$$M^H \otimes_A B = \ker f_M \otimes \text{id} \cong \ker f_M \otimes_A B = (M \otimes_A B)^H,$$