A Note on Powers of Complex Variables

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I show the case where $(z^m)^{\frac{1}{n}} = \left(z^{\frac{1}{n}}\right)^m$, where $(z^m)^{\frac{1}{n}} \neq \left(z^{\frac{1}{n}}\right)^m$, and I show a surprising result in the case of raising a complex number to an irrational power.

Let a, z, and w be non-zero complex numbers, let n and m be coprime non-zero integers, and let ρ be an irrational real number. Define

$$\operatorname{Log}(z) := \{\log|z| + i(\arg(z) + 2\pi k) : k \in \mathbb{Z}\}$$

and for each function $f : S \to \mathbb{C}$, with $S \subseteq \mathbb{C}$, define $f(S) = \{f(s) : s \in S\}$. Finally, define $z^a := \exp(a \operatorname{Log}(z))$.

I claim that $(z^m)^{\frac{1}{n}} = \left(z^{\frac{1}{n}}\right)^m$. In short, each set consists of the nth roots of z^m.

First, I claim that z^m is a singleton which consists of the element that is traditionally called z^m. This is shown as follows:

$$z^m = e^{m \operatorname{Log}(z)} = \{e^{m(\log|z| + i(\arg(z) + 2\pi k))} : k \in \mathbb{Z}\} = \{\left(|z|e^{i\arg(z)}\right)^m\} = \{z^m\}.$$

Look at the polynomial $p(x) = x^n - a$. Since

$$np(x) - xp'(x) = n(x^n - a) - x(nx^{n-1}) = -na \in \mathbb{C}^\times = \mathbb{C}[x]^\times,$$

$p(x)$ is coprime to its derivative. Therefore, $p(x)$ does not have a multiple root. On the other hand, the Fundamental Theorem of Algebra tells us that $p(x)$ has n roots, which we know to be distinct (this can be established without the fundamental theorem of algebra, by more elementary, albeit tedious methods).

Let $w \in a^{\frac{1}{n}}$. Then $w = e^{\frac{\log|a| + \arg(a)i}{n} + \frac{2\pi ik}{n}}$ for some $k \in \mathbb{Z}$. Therefore,

$$p(w) = \left(e^{\frac{\log|a| + \arg(a)i}{n} + \frac{2\pi ik}{n}}\right)^n - a = e^{\log|a| + \arg(a)i + 2\pi ik} - a = 0.$$

So $|a^{\frac{1}{n}}| \leq n$ (that is cardinality, since we are talking about sets). On the other hand, if $w, z \in a^{\frac{1}{n}}$, then $\frac{z}{w} = e^{\frac{2\pi i(k-j)}{n}}$ for some $k, j \in \mathbb{Z}$. For fixed j there are n different values depending on k, since $e^{\theta i}$ is not a positive real number for values of θ that are not integer multiples of 2π (this can be seen from the values of the sine and cosine function). So $|a^{\frac{1}{n}}| = n$.

If $w \in (z^m)^{\frac{1}{n}}$ or $w \in \left(z^{\frac{1}{n}}\right)^m$, then w is a root of $p(x)$ with $a = z^m$. On the other hand, we have already shown that $|z^{\frac{1}{n}}| = |(z^m)^{\frac{1}{n}}| = n$. If $a, w \in z^{\frac{1}{n}}$, then

$$w^m/a^m = \left(e^{\frac{2\pi i(k-j)}{n}}\right)^m = e^{\frac{2\pi im(k-j)}{n}} = 1$$

precisely when n divides $m(k-j)$. Since m, n are coprime, this is when $j \equiv k \mod n$. Therefore, $\left|\left(z^{\frac{1}{n}}\right)^m\right| = n$. In summary, we have shown that both $(z^m)^{\frac{1}{n}}$ and $\left(z^{\frac{1}{n}}\right)^m$ consist of the n roots of $p(x)$ where $a = z^m$.

On the other hand, if n, m have a common factor, then the sets are not equal. Let $z = 1$ and let $m = n = 2$. Then

$$(1^{\frac{1}{2}})^2 = \{1, -1\}^2 = \{1\},$$

but

$$(1^2)^{\frac{1}{2}} = \{1\}^{\frac{1}{2}} = \{1, -1\}.$$

Finally, I examine the case of irrational powers.

$$z^\rho = \{e^{\rho \log|z| + i\rho \arg(z)} e^{2\pi i \rho k} : k \in \mathbb{Z}\}.$$

By a standard theorem of Dirichlet (which boils down to the pigeon hole principle), the sequence $\{e^{2\pi i \rho k}\}_{k\in\mathbb{Z}}$ is dense on the unit circle. Define $R : x \mapsto x^{\frac{1}{\rho}}$, and note that $R(z^\rho) = z$. If R is continuous, then the pre-image of z under R is closed. Thus it contains a circle in $\mathbb{C}$, centered at the origin. The caveat is that R itself is multi-valued. Let us remove that by choosing a specific branch of the complex logarithm, namely that which is defined on $\mathbb{C} \setminus (-\infty, 0]$ and takes 1 to 0. Then R is a composition of continuous functions, hence continuous, but it still maps every element of z^ρ to z. Therefore, z^ρ has a circle (minus the point at which R is undefined) for its closure, centered at the origin, for every branch. Since the absolute value of the elements of z^ρ does not depend on which branch of the complex logarithm we choose, we conclude that these are all the same circle. If we would like z^ρ to be the set of $\frac{1}{\rho}$th roots of z, then it is a circle.