Chapter 4

Multiple Random Variables

Covariance and Correlation

Outline

Multivariate Distributions

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Above the n-cube from $\mathbf{y}$ to $(\mathbf{y} + \Delta\mathbf{y})$, we have probability

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where $J(\mathbf{y})$ denote the Jacobian matrix for $\mathbf{x} = g^{-1}(\mathbf{y})$. The ij-th entry in this matrix is

$$J_{ij}(\mathbf{y}) = \frac{\partial x_i}{\partial y_j}.$$

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To show this equality, we use the fact that the determinant is the unique n-linear alternating form on $n \times n$ matrices that maps the identity matrix to one.

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- (alternating) If two columns are identical, then the vectors in $\mathbf{V}$ are linearly dependent and the n-volume is 0. Swapping the columns returns the same matrix, thus $\det \mathbf{V} = -\det \mathbf{V}$. Thus $\det \mathbf{V} = 0$.

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- (linearity) Let $\tilde{\mathbf{V}}$ be the matrix resulting from the j-th column of $\mathbf{V}$ replaced by a constant times a column $i \neq j$. Then by the multilinearity of determinants,

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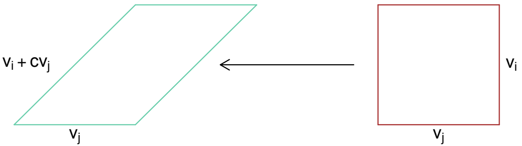

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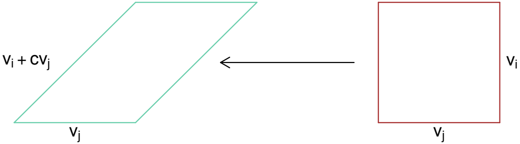

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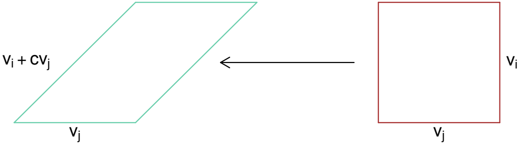

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- Thus, each of the three elementary column operations maintains equality between the volume and the absolute value of the determinant.
- Every matrix can be obtained from the identity matrix through these operations.

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Again, for A a one-to-one linear transformation and $Y = AX$, then

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$$K_{X_1+X_2+\cdots+X_d}(t) = K_{X_1}(t) + K_{X_2}(t) + \cdots + K_{X_d}(t).$$

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For $X_1, X_2, \ldots, X_2$ independent, common mean μ, variance σ^2 and sum S,

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$$\begin{aligned}\mathrm{Cov}(X_i, S) &= E[X_i S] - n\mu^2 = E[E[X_i S|S]] - n\mu^2 = E[SE[X_i|S]] - n\mu^2 = \frac{1}{n}ES^2 - n\mu^2 \\ &= \frac{1}{n}(\mathrm{Var}(S) + (ES)^2) - n\mu^2\end{aligned}$$

Covariance

- A positive covariance means that the terms $(X_1 - \mu_1)(X_2 - \mu_2)$ in the sum are more likely to be positive than negative. This occurs whenever the X_1 and X_2 variables are more often both above or below the mean in tandem than not.
- A negative covariance means that the $(X_1 - \mu_1)(X_2 - \mu_2)$ in the expectation are more likely to be negative than positive. This occurs when one of the variables is above its mean, the other is more often below.

Example. For Z_1, Z_2, bivariate standard normals,

$$\mathrm{Cov}(Z_1, Z_2) = E[Z_1 Z_2] = E[E[Z_1 Z_2|Z_1]] = E[Z_1 E[Z_2|Z_1]] = E[\rho Z_1^2] = \rho$$

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Covariance

Example. For the joint density example,

$$EX_1X_2 = \frac{4}{5}\int_0^1\int_0^1 x_1x_2(x_1+x_2+x_1x_2)\,dx_2\,dx_1 = \frac{4}{5}\int_0^1\int_0^1 (x_1^2x_2+x_1x_2^2+x_1^2x_2^2)\,dx_2\,dx_1$$

Covariance

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\begin{aligned}
EX_1X_2 &= \frac{4}{5}\int_0^1\int_0^1 x_1x_2(x_1+x_2+x_1x_2)\,dx_2\,dx_1 = \frac{4}{5}\int_0^1\int_0^1 (x_1^2x_2+x_1x_2^2+x_1^2x_2^2)\,dx_2\,dx_1 \\
&= \frac{4}{5}\int_0^1 \left(\frac{1}{2}x_1^2x_2^2+\frac{1}{3}x_1x_2^3+\frac{1}{3}x_1^2x_2^3\right)\Big|_0^1\,dx_1
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&= \frac{4}{5}\int_0^1 \left(\frac{1}{2}x_1^2x_2^2+\frac{1}{3}x_1x_2^3+\frac{1}{3}x_1^2x_2^3\right)\Big|_0^1\,dx_1 = \frac{4}{5}\int_0^1\left(\frac{5}{6}x_1^2+\frac{1}{3}x_1\right)\,dx_1
\end{aligned}
$$

Covariance

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\begin{aligned}
EX_1X_2 &= \frac{4}{5}\int_0^1\int_0^1 x_1x_2(x_1+x_2+x_1x_2)\,dx_2\,dx_1 = \frac{4}{5}\int_0^1\int_0^1 (x_1^2x_2+x_1x_2^2+x_1^2x_2^2)\,dx_2\,dx_1 \\
&= \frac{4}{5}\int_0^1 \left(\frac{1}{2}x_1^2x_2^2+\frac{1}{3}x_1x_2^3+\frac{1}{3}x_1^2x_2^3\right)\Big|_0^1 dx_1 = \frac{4}{5}\int_0^1 \left(\frac{5}{6}x_1^2+\frac{1}{3}x_1\right)dx_1 \\
&= \frac{4}{5}\left(\frac{5}{18}x_1^3+\frac{1}{6}x_1^2\right)\Big|_0^1
\end{aligned}
$$

Covariance

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$$\begin{aligned} EX_1X_2 &= \frac{4}{5}\int_0^1\int_0^1 x_1x_2(x_1+x_2+x_1x_2)\,dx_2\,dx_1 = \frac{4}{5}\int_0^1\int_0^1 (x_1^2x_2+x_1x_2^2+x_1^2x_2^2)\,dx_2\,dx_1 \\ &= \frac{4}{5}\int_0^1 \left(\frac{1}{2}x_1^2x_2^2+\frac{1}{3}x_1x_2^3+\frac{1}{3}x_1^2x_2^3\right)\Big|_0^1 dx_1 = \frac{4}{5}\int_0^1\left(\frac{5}{6}x_1^2+\frac{1}{3}x_1\right)\,dx_1 \\ &= \frac{4}{5}\left(\frac{5}{18}x_1^3+\frac{1}{6}x_1^2\right)\Big|_0^1 = \frac{4}{5}\left(\frac{5}{18}+\frac{1}{6}\right) \end{aligned}$$

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&= \frac{4}{5}\int_0^1 \left(\frac{1}{2}x_1^2x_2^2+\frac{1}{3}x_1x_2^3+\frac{1}{3}x_1^2x_2^3\right)\Big|_0^1\,dx_1 = \frac{4}{5}\int_0^1\left(\frac{5}{6}x_1^2+\frac{1}{3}x_1\right)\,dx_1 \\
&= \frac{4}{5}\left(\frac{5}{18}x_1^3+\frac{1}{6}x_1^2\right)\Big|_0^1 = \frac{4}{5}\left(\frac{5}{18}+\frac{1}{6}\right) = \frac{16}{45}
\end{aligned}
$$

Covariance

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$$\begin{aligned} EX_1X_2 &= \frac{4}{5}\int_0^1\int_0^1 x_1x_2(x_1+x_2+x_1x_2)\,dx_2\,dx_1 = \frac{4}{5}\int_0^1\int_0^1 (x_1^2x_2+x_1x_2^2+x_1^2x_2^2)\,dx_2\,dx_1 \\ &= \frac{4}{5}\int_0^1 \left(\frac{1}{2}x_1^2x_2^2+\frac{1}{3}x_1x_2^3+\frac{1}{3}x_1^2x_2^3\right)\Big|_0^1\,dx_1 = \frac{4}{5}\int_0^1\left(\frac{5}{6}x_1^2+\frac{1}{3}x_1\right)\,dx_1 \\ &= \frac{4}{5}\left(\frac{5}{18}x_1^3+\frac{1}{6}x_1^2\right)\Big|_0^1 = \frac{4}{5}\left(\frac{5}{18}+\frac{1}{6}\right) = \frac{16}{45} \end{aligned}$$

$$EX_1 = EX_2$$

Covariance

Example. For the joint density example,

$$
\begin{aligned}
EX_1X_2 &= \frac{4}{5}\int_0^1\int_0^1 x_1x_2(x_1+x_2+x_1x_2)\,dx_2\,dx_1 = \frac{4}{5}\int_0^1\int_0^1 (x_1^2x_2+x_1x_2^2+x_1^2x_2^2)\,dx_2\,dx_1 \\
&= \frac{4}{5}\int_0^1 \left(\frac{1}{2}x_1^2x_2^2+\frac{1}{3}x_1x_2^3+\frac{1}{3}x_1^2x_2^3\right)\Big|_0^1 dx_1 = \frac{4}{5}\int_0^1\left(\frac{5}{6}x_1^2+\frac{1}{3}x_1\right)\,dx_1 \\
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\end{aligned}
$$

$$EX_1 = EX_2 = \frac{2}{5}\int_0^1 x_1(3x_1+1)\,dx_1 =$$

Covariance

Example. For the joint density example,

$$
\begin{aligned}
EX_1X_2 &= \frac{4}{5}\int_0^1\int_0^1 x_1x_2(x_1+x_2+x_1x_2)\,dx_2\,dx_1 = \frac{4}{5}\int_0^1\int_0^1 (x_1^2x_2+x_1x_2^2+x_1^2x_2^2)\,dx_2\,dx_1 \\
&= \frac{4}{5}\int_0^1 \left(\frac{1}{2}x_1^2x_2^2+\frac{1}{3}x_1x_2^3+\frac{1}{3}x_1^2x_2^3\right)\Big|_0^1 dx_1 = \frac{4}{5}\int_0^1 \left(\frac{5}{6}x_1^2+\frac{1}{3}x_1\right)\,dx_1 \\
&= \frac{4}{5}\left(\frac{5}{18}x_1^3+\frac{1}{6}x_1^2\right)\Big|_0^1 = \frac{4}{5}\left(\frac{5}{18}+\frac{1}{6}\right) = \frac{16}{45}
\end{aligned}
$$

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EX_1 = EX_2 = \frac{2}{5}\int_0^1 x_1(3x_1+1)\,dx_1 = \frac{2}{5}\left(x_1^3+\frac{1}{2}x_1^2\right)\Big|_0^1
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Covariance

Example. For the joint density example,

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Thus,

$$\mathrm{Cov}(X_1, X_2) =$$

Covariance

Example. For the joint density example,

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EX_1X_2 &= \frac{4}{5}\int_0^1\int_0^1 x_1x_2(x_1+x_2+x_1x_2)\,dx_2\,dx_1 = \frac{4}{5}\int_0^1\int_0^1 (x_1^2x_2+x_1x_2^2+x_1^2x_2^2)\,dx_2\,dx_1 \\
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&= \frac{4}{5}\left(\frac{5}{18}x_1^3+\frac{1}{6}x_1^2\right)\Big|_0^1 = \frac{4}{5}\left(\frac{5}{18}+\frac{1}{6}\right) = \frac{16}{45}
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Covariance

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$$EX_1 = EX_2 = \frac{2}{5}\int_0^1 x_1(3x_1+1)\,dx_1 = \frac{2}{5}\left(x_1^3+\frac{1}{2}x_1^2\right)\Big|_0^1 = \frac{2}{5}\cdot\frac{3}{2} = \frac{3}{5}.$$

Thus,

$$\mathrm{Cov}(X_1,X_2) = \frac{16}{45} - \left(\frac{3}{5}\right)^2$$

Covariance

Example. For the joint density example,

$$\begin{aligned} EX_1X_2 &= \frac{4}{5}\int_0^1\int_0^1 x_1x_2(x_1+x_2+x_1x_2)\,dx_2\,dx_1 = \frac{4}{5}\int_0^1\int_0^1 (x_1^2x_2+x_1x_2^2+x_1^2x_2^2)\,dx_2\,dx_1 \\ &= \frac{4}{5}\int_0^1 \left(\frac{1}{2}x_1^2x_2^2+\frac{1}{3}x_1x_2^3+\frac{1}{3}x_1^2x_2^3\right)\Big|_0^1 dx_1 = \frac{4}{5}\int_0^1 \left(\frac{5}{6}x_1^2+\frac{1}{3}x_1\right)\,dx_1 \\ &= \frac{4}{5}\left(\frac{5}{18}x_1^3+\frac{1}{6}x_1^2\right)\Big|_0^1 = \frac{4}{5}\left(\frac{5}{18}+\frac{1}{6}\right) = \frac{16}{45} \end{aligned}$$

$$EX_1 = EX_2 = \frac{2}{5}\int_0^1 x_1(3x_1+1)\,dx_1 = \frac{2}{5}\left(x_1^3+\frac{1}{2}x_1^2\right)\Big|_0^1 = \frac{2}{5}\cdot\frac{3}{2} = \frac{3}{5}.$$

Thus,

$$\mathrm{Cov}(X_1,X_2) = \frac{16}{45}-\left(\frac{3}{5}\right)^2 = \frac{80-81}{225} = -\frac{1}{225}.$$

Correlation

Covariance fails to take into account the scale of the measurements - larger values lead to larger covariance.

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$$\sigma_1^2 = \sigma_2^2 = \frac{2}{5}\int_0^1 x^2(3x+1)\,dx - \left(\frac{3}{5}\right)^2$$

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and

$$\rho_{X_1,X_2} = \frac{-1/225}{11/150}$$

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In the example,

$$\sigma_1^2 = \sigma_2^2 = \frac{2}{5}\int_0^1 x^2(3x+1)\,dx - \left(\frac{3}{5}\right)^2 = \frac{2}{5}\cdot\frac{13}{12} - \frac{9}{25} = \frac{11}{150}.$$

and

$$\rho_{X_1,X_2} = \frac{-1/225}{11/150} = -\frac{2}{33} = -0.06.$$

Covariance

Exercise. $\mathrm{Cov}(a_1 + b_1 X_1, a_2 + b_2 X_2) = b_1 b_2 \mathrm{Cov}(X_1, X_2)$

Covariance

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$$\text{Cov}(a_1 + b_1X_1, a_2 + b_2X_2) = E[((a_1 + b_1X_1) - (a_1 + b_1\mu_1))((a_2 + b_2X_2) - (a_2 + b_2\mu_2))]$$

Covariance

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$$\begin{aligned}\text{Cov}(a_1 + b_1X_1, a_2 + b_2X_2) &= E[((a_1 + b_1X_1) - (a_1 + b_1\mu_1))((a_2 + b_2X_2) - (a_2 + b_2\mu_2))] \\ &= E[(b_1(X_1 - \mu_1))(b_2(X_2 - \mu_2))]\end{aligned}$$

Covariance

Exercise. $\text{Cov}(a_1 + b_1X_1, a_2 + b_2X_2) = b_1b_2\text{Cov}(X_1, X_2)$

$$\begin{aligned}\text{Cov}(a_1 + b_1X_1, a_2 + b_2X_2) &= E[((a_1 + b_1X_1) - (a_1 + b_1\mu_1))((a_2 + b_2X_2) - (a_2 + b_2\mu_2))] \\ &= E[(b_1(X_1 - \mu_1))(b_2(X_2 - \mu_2))] \\ &= b_1b_2E[(X_1 - \mu_1)(X_2 - \mu_2)\end{aligned}$$

Covariance

Exercise. $\mathrm{Cov}(a_1 + b_1 X_1, a_2 + b_2 X_2) = b_1 b_2 \mathrm{Cov}(X_1, X_2)$

$$\begin{aligned}
\mathrm{Cov}(a_1 + b_1 X_1, a_2 + b_2 X_2) &= E[((a_1 + b_1 X_1) - (a_1 + b_1 \mu_1))((a_2 + b_2 X_2) - (a_2 + b_2 \mu_2))] \\
&= E[(b_1(X_1 - \mu_1))(b_2(X_2 - \mu_2))] \\
&= b_1 b_2 E[(X_1 - \mu_1)(X_2 - \mu_2) = b_1 b_2 \mathrm{Cov}(X_1, X_2)
\end{aligned}$$

Covariance

Exercise. $\text{Cov}(a_1 + b_1 X_1, a_2 + b_2 X_2) = b_1 b_2 \text{Cov}(X_1, X_2)$

$$\begin{aligned}\text{Cov}(a_1 + b_1 X_1, a_2 + b_2 X_2) &= E[((a_1 + b_1 X_1) - (a_1 + b_1 \mu_1))((a_2 + b_2 X_2) - (a_2 + b_2 \mu_2))] \\ &= E[(b_1(X_1 - \mu_1))(b_2(X_2 - \mu_2))] \\ &= b_1 b_2 E[(X_1 - \mu_1)(X_2 - \mu_2) = b_1 b_2 \text{Cov}(X_1, X_2)\end{aligned}$$

Continuing, note that $\text{Cov}(X_i, X_i) = \text{Var}(X_i)$

Covariance

Exercise. $\mathrm{Cov}(a_1 + b_1 X_1, a_2 + b_2 X_2) = b_1 b_2 \mathrm{Cov}(X_1, X_2)$

$$\begin{aligned}
\mathrm{Cov}(a_1 + b_1 X_1, a_2 + b_2 X_2) &= E[((a_1 + b_1 X_1) - (a_1 + b_1 \mu_1))((a_2 + b_2 X_2) - (a_2 + b_2 \mu_2))] \\
&= E[(b_1(X_1 - \mu_1))(b_2(X_2 - \mu_2))] \\
&= b_1 b_2 E[(X_1 - \mu_1)(X_2 - \mu_2) = b_1 b_2 \mathrm{Cov}(X_1, X_2)
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Continuing, note that $\mathrm{Cov}(X_i, X_i) = \mathrm{Var}(X_i)$

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Exercise. $\mathrm{Cov}(a_1 + b_1 X_1, a_2 + b_2 X_2) = b_1 b_2 \mathrm{Cov}(X_1, X_2)$

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\mathrm{Cov}(a_1 + b_1 X_1, a_2 + b_2 X_2) &= E[((a_1 + b_1 X_1) - (a_1 + b_1 \mu_1))((a_2 + b_2 X_2) - (a_2 + b_2 \mu_2))] \\
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In particular,

$$0 \leq \sigma^2_{X_1 + cX_2} = \sigma_1^2 + 2c\rho_{X_1,X_2}\sigma_1\sigma_2 + \sigma_2^2 c^2.$$

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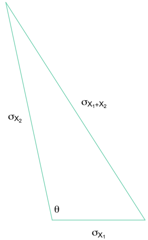

Figure: For the law of cosines, let $a = \sigma_{X_1}$, $b = \sigma_{X_2}$, $\sigma_{X_1+X_2}$ and $r = -\cos\theta$

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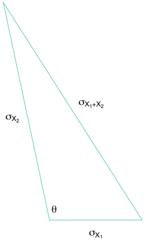

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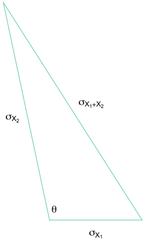

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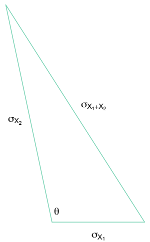

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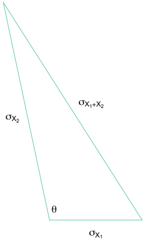

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$$\begin{aligned}\mathrm{Var}(X) &= \sum_{i=1}^{k}\sum_{j=1}^{k}\mathrm{Cov}(X_i,X_j) = \sum_{i=1}^{k}\mathrm{Var}(X_i) + \sum_{i=1}^{k}\sum_{j\neq i}\mathrm{Cov}(X_i,X_j) \\ &= k\frac{mn}{(m+n)^2} + k(k-1)\left(\frac{-mn}{(m+n)^2(m+n-1)}\right)\end{aligned}$$

Hypergeometric Random Variables

The correlation

$$\rho_{X_i,X_j} = \frac{\mathrm{Cov}(X_i,X_j)}{\mathrm{Var}(X_i)} = \frac{-mn}{(m+n)^2(m+n-1)} \Big/ \frac{mn}{(m+n)^2} = -\frac{1}{m+n-1}$$

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&= k\frac{mn}{(m+n)^2}\left(1 - \frac{k-1}{m+n-1}\right) = kp(1-p)\frac{N-k}{N-1}
\end{aligned}$$

Hypergeometric Random Variables

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where $N = m + n$ is the total number of balls

Hypergeometric Random Variables

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\end{aligned}$$

where $N = m+n$ is the total number of balls and $p = m/(m+n)$ is the probability that a white ball is chosen.