Chapter 4
Multiple Random Variables
Covariance and Correlation
Outline

Multivariate Distributions
  Multivariate Normal Random Variables
  Sums of Independent Random Variable

Covariance

Covariance
  Correlation
  Hypergeometric Random Variable
Multivariate Distributions

Many of the facts about bivariate distributions have straightforward generalizations to the general multivariate case.

For a $d$-dimensional discrete random variable $X = (X_1, X_2, \ldots, X_d)$, take $x \in \mathbb{R}^d$, we have the probability mass function $f_X(x) = P\{X = x\}$. 
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$g : S \rightarrow \mathbb{R}^d$, a one-to-one mapping, write $Y = g(X)$. 

$g^{-1}(y + \Delta y) \approx g^{-1}(y) + \nabla g^{-1}(y) \cdot \Delta y = x + J(y) \cdot \Delta y$, 

where $J(y)$ denote the Jacobian matrix for $x = g^{-1}(y)$. The $ij$-th entry in this matrix is $J_{ij}(y) = \frac{\partial x_i}{\partial y_j}$. 

Above the $n$-cube from $y$ to $(y + \Delta y)$, we have probability $f_Y(y)(\Delta y) \approx P\{y < Y \leq y + \Delta y\}$

For $x = g^{-1}(y)$, this probability is equal to the area of image of the $n$-cube from $y$ to $y + \Delta y$ under the map $g^{-1}$ times the density $f_X(x)$. 

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To show this equality, we use the fact that the determinant is the unique \( n \)-linear alternating form on \( n \times n \) matrices that maps the identity matrix to one.
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- *(linearity)* Multiplying a column by a non-zero constant \( c \) results in change in the determinant by a factor of \( c \) and the volume by a factor \(|c|\).
- *(alternating)* If two columns are identical, then the vectors in \( \mathbf{V} \) are linearly dependent and the \( n \)-volume is 0. Swapping the columns returns the same matrix, thus \( \det \mathbf{V} = -\det \mathbf{V} \). Thus \( \det \mathbf{V} = 0 \).
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- (linearity) Let $\tilde{V}$ be the matrix resulting from the $j$-th column of $V$ replaced by a constant times a column $i \neq j$. Then by the multilinearity of determinants,

$$\det(\tilde{V}) = \det V + \det(\tilde{V} - V) = \det V + 0$$
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• Thus, each of the three elementary column operations maintains equality between the volume and the absolute value of the determinant.

• Every matrix can be obtained from the identity matrix through these operations.
Multivariate Normal Random Variables

Again, for $A$ a one-to-one linear transformation and $Y = AX$, then

$$f_Y(y) = f_X(A^{-1}y) |\det(A^{-1})| = \frac{1}{|\det(A)|} f_X(A^{-1}y).$$
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For $Z_1, \ldots, Z_n$ independent $N(0, 1)$, the density

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{z_1^2}{2} \right) \cdots \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{z_n^2}{2} \right).$$
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Any random vector $Y = AZ$ obtained as the linear transformation is called a centered multivariate normal random variable.
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Multivariate Distributions

Covariance

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Sums of Independent Random Variable

- Random variables $X_1, X_2, \ldots, X_d$ are independent provided that for any choice of sets $B_1, B_2, \ldots, B_d$,
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$$P\{X_1 \in B_1, X_2 \in B_2, \ldots, X_d \in B_d\}$$
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For independent random variables
- For either mass functions or density functions, the joint mass or density function is the product of the one-dimensional marginals.
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$$f_X(x) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_d}(x_d).$$
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$$E[g_1(X_1)g_2(X_2) \cdots g_d(X_d)]$$
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$$f_X(x) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_d}(x_d).$$

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$$E[g_1(X_1)g_2(X_2) \cdots g_d(X_d)] = E[g_1(X_1)]E[g_2(X_2)] \cdots E[g_d(X_d)]$$

provided each of these expectations exist.
Sums of Independent Random Variable

- For non-negative integer-valued variables, the probability generating function of the sum is the product of one-dimensional probability generating functions.
Sums of Independent Random Variable

- For non-negative integer-valued variables, the probability generating function of the sum is the product of one-dimensional probability generating functions.

\[ \rho_{X_1 + X_2 + \cdots + X_d}(z) \]
Sums of Independent Random Variable

- For non-negative integer-valued variables, the probability generating function of the sum is the product of one-dimensional probability generating functions.

\[ \rho_{X_1+X_2+...+X_d}(z) = \rho_{X_1}(z)\rho_{X_2}(z)\cdots\rho_{X_d}(z). \]
Sums of Independent Random Variable

• For non-negative integer-valued variables, the probability generating function of the sum is the product of one-dimensional probability generating functions.

\[ \rho_{X_1+X_2+\ldots+X_d}(z) = \rho_{X_1}(z)\rho_{X_2}(z)\cdots\rho_{X_d}(z). \]

• For any random variables, the mass or density function of the sum is the convolution of one-dimensional probability probability masses or densities, respectively.

\[ f_{X_1+X_2+\ldots+X_d}(x) = f_{X_1}(x)\ast f_{X_2}(x)\ast\cdots\ast f_{X_d}(x). \]
Sums of Independent Random Variable

- For non-negative integer-valued variables, the probability generating function of the sum is the product of one-dimensional probability generating functions.

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\[ f_{X_1 + X_2 + \ldots + X_d}(x) = f_{X_1}(x) * f_{X_2}(x) * \cdots * f_{X_d}(x). \]
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\[ \rho_{X_1+X_2+\ldots+X_d}(z) = \rho_{X_1}(z)\rho_{X_2}(z) \cdots \rho_{X_d}(z). \]

• For any random variables, the mass or density function of the sum is the convolution of one-dimensional probability masses or densities, respectively.

\[ f_{X_1+X_2+\ldots+X_d}(x) = f_{X_1}(x) * f_{X_2}(x) * \cdots * f_{X_d}(x). \]

• For any random variables, the moment generating function of the sum is the product of one-dimensional probability generating functions.
Sums of Independent Random Variable

- For non-negative integer-valued variables, the probability generating function of the sum is the product of one-dimensional probability generating functions.

\[ \rho_{X_1+X_2+\cdots+X_d}(z) = \rho_{X_1}(z)\rho_{X_2}(z)\cdots\rho_{X_d}(z). \]

- For any random variables, the mass or density function of the sum is the convolution of one-dimensional probability masses or densities, respectively.

\[ f_{X_1+X_2+\cdots+X_d}(x) = f_{X_1}(x) \ast f_{X_2}(x) \ast \cdots \ast f_{X_d}(x). \]

- For any random variables, the moment generating function of the sum is the product of one-dimensional probability generating functions.

\[ M_{X_1+X_2+\cdots+X_d}(t) = M_{X_1}(t)M_{X_2}(t)\cdots M_{X_d}(t). \]
Sums of Independent Random Variable

- For non-negative integer-valued variables, the probability generating function of the sum is the product of one-dimensional probability generating functions.

\[ \rho_{X_1+X_2+\ldots+X_d}(z) = \rho_{X_1}(z)\rho_{X_2}(z)\cdots\rho_{X_d}(z). \]

- For any random variables, the mass or density function of the sum is the convolution of one-dimensional probability masses or densities, respectively.

\[ f_{X_1+X_2+\ldots+X_d}(x) = f_{X_1}(x) * f_{X_2}(x) * \cdots * f_{X_d}(x). \]

- For any random variables, the moment generating function of the sum is the product of one-dimensional probability generating functions.

\[ M_{X_1+X_2+\ldots+X_d}(t) = M_{X_1}(t)M_{X_2}(t)\cdots M_{X_d}(t). \]

- For any random variables, the cumulant generating function of the sum is the sum of one-dimensional cumulant generating functions.
Sums of Independent Random Variable

• For non-negative integer-valued variables, the probability generating function of the sum is the product of one-dimensional probability generating functions.

\[ \rho_{X_1+X_2+\ldots+X_d}(z) = \rho_{X_1}(z) \rho_{X_2}(z) \cdots \rho_{X_d}(z). \]

• For any random variables, the mass or density function of the sum is the convolution of one-dimensional probability masses or densities, respectively.

\[ f_{X_1+X_2+\ldots+X_d}(x) = f_{X_1}(x) * f_{X_2}(x) * \cdots * f_{X_d}(x). \]

• For any random variables, the moment generating function of the sum is the product of one-dimensional probability generating functions.

\[ M_{X_1+X_2+\ldots+X_d}(t) = M_{X_1}(t) M_{X_2}(t) \cdots M_{X_d}(t). \]

• For any random variables, the cumulant generating function of the sum is the sum of one-dimensional cumulant generating functions.

\[ K_{X_1+X_2+\ldots+X_d}(t) = K_{X_1}(t) + K_{X_2}(t) + \cdots + K_{X_d}(t). \]
Covariance

Here, we shall assume that the random variables under consideration have positive and finite variance.
Covariance

Here, we shall assume that the random variables under consideration have positive and finite variance.

One simple way to assess the relationship between two random variables $X_1$ and $X_2$ with respective means $\mu_1$ and $\mu_2$ is to compute their covariance.
Covariance

Here, we shall assume that the random variables under consideration have positive and finite variance.

One simple way to assess the relationship between two random variables $X_1$ and $X_2$ with respective means $\mu_1$ and $\mu_2$ is to compute their covariance.

$$\text{Cov}(X_1, X_2) = E[(X_1 - \mu_1)(X_2 - \mu_2)]$$
Here, we shall assume that the random variables under consideration have positive and finite variance.

One simple way to assess the relationship between two random variables $X_1$ and $X_2$ with respective means $\mu_1$ and $\mu_2$ is to compute their covariance.

\[
\text{Cov}(X_1, X_2) = E[(X_1 - \mu_1)(X_2 - \mu_2)] = E[X_1 X_2] - \mu_2 E X_1 - \mu_1 E X_2 + \mu_1 \mu_2
\]
Covariance

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One simple way to assess the relationship between two random variables \( X_1 \) and \( X_2 \) with respective means \( \mu_1 \) and \( \mu_2 \) is to compute their covariance.

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\text{Cov}(X_1, X_2) = E[(X_1 - \mu_1)(X_2 - \mu_2)] \\
= E[X_1 X_2] - \mu_2 EX_1 - \mu_1 EX_2 + \mu_1 \mu_2 \\
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Covariance

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One simple way to assess the relationship between two random variables \(X_1\) and \(X_2\) with respective means \(\mu_1\) and \(\mu_2\) is to compute their covariance.

\[
\text{Cov}(X_1, X_2) = \mathbb{E}[(X_1 - \mu_1)(X_2 - \mu_2)]
\]
\[
= \mathbb{E}[X_1X_2] - \mu_2\mathbb{E}X_1 - \mu_1\mathbb{E}X_2 + \mu_1\mu_2
\]
\[
= \mathbb{E}[X_1X_2] - \mu_1\mu_2
\]

Exercise. If \(X_1\) and \(X_2\) are independent then \(\text{Cov}(X_1, X_2) = 0\)
Covariance

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One simple way to assess the relationship between two random variables $X_1$ and $X_2$ with respective means $\mu_1$ and $\mu_2$ is to compute their covariance.

$$\text{Cov}(X_1, X_2) = E[(X_1 - \mu_1)(X_2 - \mu_2)]$$
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$$E[(X_1 - \mu_1)(X_2 - \mu_2)] =$$
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$$E[(X_1 - \mu_1)(X_2 - \mu_2)] = E[X_1 - \mu_1]E[X_2 - \mu_2] =$$
Covariance

Here, we shall assume that the random variables under consideration have positive and finite variance.

One simple way to assess the relationship between two random variables $X_1$ and $X_2$ with respective means $\mu_1$ and $\mu_2$ is to compute their covariance.

$$\text{Cov}(X_1, X_2) = E[(X_1 - \mu_1)(X_2 - \mu_2)]$$

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Exercise. If $X_1$ and $X_2$ are independent then $\text{Cov}(X_1, X_2) = 0$

$$E[(X_1 - \mu_1)(X_2 - \mu_2)] = E[X_1 - \mu_1]E[X_2 - \mu_2] = 0 \cdot 0 = 0.$$
Covariance

- A positive covariance means that the terms \((X_1 - \mu_1)(X_2 - \mu_2)\) in the sum are more likely to be positive than negative. This occurs whenever the \(X_1\) and \(X_2\) variables are more often both above or below the mean in tandem than not.
A positive covariance means that the terms \((X_1 - \mu_1)(X_2 - \mu_2)\) in the sum are more likely to be positive than negative. This occurs whenever the \(X_1\) and \(X_2\) variables are more often both above or below the mean in tandem than not.

A negative covariance means that the \((X_1 - \mu_1)(X_2 - \mu_2)\) in the expectation are more likely to be negative than positive. This occurs when one of the variables is above its mean, the other is more often below.
Covariance

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Example. For \(Z_1, Z_2\), bivariate standard normals,

\[
\text{Cov}(Z_1, Z_2)
\]
Covariance

- A **positive covariance** means that the terms $(X_1 - \mu_1)(X_2 - \mu_2)$ in the sum are more likely to be positive than negative. This occurs whenever the $X_1$ and $X_2$ variables are more often both above or below the mean in tandem than not.

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**Example.** For $Z_1, Z_2$, bivariate standard normals,

\[
\text{Cov}(Z_1, Z_2) = E[Z_1Z_2] =
\]
Covariance

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**Example.** For \(Z_1, Z_2\), bivariate standard normals,

\[
\text{Cov}(Z_1, Z_2) = E[Z_1Z_2] = E[E[Z_1Z_2|Z_1]] =
\]
Covariance

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\[
\text{Cov}(Z_1, Z_2) = E[Z_1Z_2] = E[E[Z_1Z_2|Z_1]] = E[Z_1E[Z_2|Z_1]] = E[\rho Z_1^2] = \rho
\]
Covariance

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**Example.** For \(Z_1, Z_2\), bivariate standard normals,

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\text{Cov}(Z_1, Z_2) = E[Z_1Z_2] = E[E[Z_1Z_2|Z_1]] = E[Z_1E[Z_2|Z_1]] = E[\rho Z_1^2] = \rho
\]

For \(X_1, X_2, \ldots, X_2\) independent, common mean \(\mu\), variance \(\sigma^2\) and sum \(S\),

\[
\text{Cov}(X_i, S) = E[X_iS] - n\mu^2
\]
Covariance

- A **positive covariance** means that the terms \((X_1 - \mu_1)(X_2 - \mu_2)\) in the sum are more likely to be positive than negative. This occurs whenever the \(X_1\) and \(X_2\) variables are more often both above or below the mean in tandem than not.
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**Example.** For \(Z_1, Z_2\), bivariate standard normals,

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\text{Cov}(Z_1, Z_2) = E[Z_1Z_2] = E[E[Z_1Z_2|Z_1]] = E[Z_1E[Z_2|Z_1]] = E[\rho Z_1^2] = \rho
\]

For \(X_1, X_2, \ldots, X_2\) independent, common mean \(\mu\), variance \(\sigma^2\) and sum \(S\),

\[
\text{Cov}(X_i, S) = E[X_iS] - n\mu^2 = E[E[X_iS|S]] - n\mu^2
\]
Covariance

- A positive covariance means that the terms \((X_1 - \mu_1)(X_2 - \mu_2)\) in the sum are more likely to be positive than negative. This occurs whenever the \(X_1\) and \(X_2\) variables are more often both above or below the mean in tandem than not.
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Example. For \(Z_1, Z_2\), bivariate standard normals,

\[
\text{Cov}(Z_1, Z_2) = E[Z_1Z_2] = E[E[Z_1Z_2|Z_1]] = E[Z_1E[Z_2|Z_1]] = E[\rho Z_1^2] = \rho
\]

For \(X_1, X_2, \ldots, X_2\) independent, common mean \(\mu\), variance \(\sigma^2\) and sum \(S\),

\[
\text{Cov}(X_i, S) = E[X_iS] - n\mu^2 = E[E[X_iS|S]] - n\mu^2 = E[SE[X_i|S]] - n\mu^2
\]
Covariance

- A positive covariance means that the terms \((X_1 - \mu_1)(X_2 - \mu_2)\) in the sum are more likely to be positive than negative. This occurs whenever the \(X_1\) and \(X_2\) variables are more often both above or below the mean in tandem than not.

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Example. For \(Z_1, Z_2\), bivariate standard normals,

\[
\text{Cov}(Z_1, Z_2) = E[Z_1Z_2] = E[E[Z_1Z_2|Z_1]] = E[Z_1E[Z_2|Z_1]] = E[\rho Z_1^2] = \rho
\]

For \(X_1, X_2, \ldots, X_2\) independent, common mean \(\mu\), variance \(\sigma^2\) and sum \(S\),

\[
\text{Cov}(X_i, S) = E[X_iS] - n\mu^2 = E[E[X_iS|S]] - n\mu^2 = E[SE[X_i|S]] - n\mu^2 = \frac{1}{n}ES^2 - n\mu^2
\]
Covariance

- A **positive covariance** means that the terms \((X_1 - \mu_1)(X_2 - \mu_2)\) in the sum are more likely to be positive than negative. This occurs whenever the \(X_1\) and \(X_2\) variables are more often both above or below the mean in tandem than not.
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**Example.** For \(Z_1, Z_2\), bivariate standard normals,

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\text{Cov}(Z_1, Z_2) = E[Z_1 Z_2] = E[E[Z_1 Z_2|Z_1]] = E[Z_1 E[Z_2|Z_1]] = E[\rho Z_1^2] = \rho
\]

For \(X_1, X_2, \ldots, X_2\) independent, common mean \(\mu\), variance \(\sigma^2\) and sum \(S\),

\[
\text{Cov}(X_i, S) = E[X_i S] - n\mu^2 = E[E[X_i S|S]] - n\mu^2 = E[SE[X_i|S]] - n\mu^2 = \frac{1}{n}ES^2 - n\mu^2
\]

\[
= \frac{1}{n}(\text{Var}(S) + (ES)^2) - n\mu^2
\]
Covariance

- A positive covariance means that the terms \((X_1 - \mu_1)(X_2 - \mu_2)\) in the sum are more likely to be positive than negative. This occurs whenever the \(X_1\) and \(X_2\) variables are more often both above or below the mean in tandem than not.

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\text{Cov}(Z_1, Z_2) = E[Z_1Z_2] = E[E[Z_1Z_2|Z_1]] = E[Z_1E[Z_2|Z_1]] = E[\rho Z_1^2] = \rho
\]

For \(X_1, X_2, \ldots, X_2\) independent, common mean \(\mu\), variance \(\sigma^2\) and sum \(S\),

\[
\text{Cov}(X_i, S) = E[X_i S] - n\mu^2 = E[E[X_i S|S]] - n\mu^2 = E[SE[X_i|S]] - n\mu^2 = \frac{1}{n}ES^2 - n\mu^2
\]

\[
= \frac{1}{n}(\text{Var}(S) + (ES)^2) - n\mu^2 = \frac{1}{n}(n\sigma^2 + (n\mu)^2) - n\mu^2
\]
Covariance

- A positive covariance means that the terms \((X_1 - \mu_1)(X_2 - \mu_2)\) in the sum are more likely to be positive than negative. This occurs whenever the \(X_1\) and \(X_2\) variables are more often both above or below the mean in tandem than not.

- A negative covariance means that the \((X_1 - \mu_1)(X_2 - \mu_2)\) in the expectation are more likely to be negative than positive. This occurs when one of the variables is above its mean, the other is more often below.

Example. For \(Z_1, Z_2\), bivariate standard normals,

\[
\text{Cov}(Z_1, Z_2) = E[Z_1Z_2] = E[E[Z_1Z_2|Z_1]] = E[Z_1E[Z_2|Z_1]] = E[\rho Z_1^2] = \rho
\]

For \(X_1, X_2, \ldots, X_2\) independent, common mean \(\mu\), variance \(\sigma^2\) and sum \(S\),

\[
\text{Cov}(X_i, S) = E[X_iS] - n\mu^2 = E[E[X_iS|S]] - n\mu^2 = E[SE[X_i|S]] - n\mu^2 = \frac{1}{n}ES^2 - n\mu^2
\]

\[= \frac{1}{n}(\text{Var}(S) + (ES)^2) - n\mu^2 = \frac{1}{n}(n\sigma^2 + (n\mu)^2) - n\mu^2 = \sigma^2\]
Covariance

Example. For the joint density example,

\[ EX_1 X_2 = \frac{4}{5} \int_0^1 \int_0^1 x_1 x_2 (x_1 + x_2 + x_1 x_2) \, dx_2 \, dx_1 = \frac{4}{5} \int_0^1 \int_0^1 (x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_2^2) \, dx_2 \, dx_1 \]
Example. For the joint density example,

\[
EX_1X_2 = \frac{4}{5} \int_0^1 \int_0^1 x_1 x_2 (x_1 + x_2 + x_1 x_2) \, dx_2 \, dx_1 = \frac{4}{5} \int_0^1 \int_0^1 (x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_2^2) \, dx_2 \, dx_1
\]

\[
= \frac{4}{5} \int_0^1 \left( \frac{1}{2} x_1^2 x_2^2 + \frac{1}{3} x_1 x_2^3 + \frac{1}{3} x_1^2 x_2^3 \right) \bigg|_0^1 \, dx_1
\]

Thus, \( \text{Cov}(X_1, X_2) = \frac{16}{45} - \left(\frac{3}{5}\right)^2 = \frac{80}{45} - \frac{81}{225} = -\frac{1}{225}. \)
Covariance

Example. For the joint density example,

\[ \text{EX}_1X_2 = \frac{4}{5} \int_0^1 \int_0^1 x_1x_2(x_1 + x_2 + x_1x_2) \, dx_2 \, dx_1 = \frac{4}{5} \int_0^1 \int_0^1 (x_1^2x_2 + x_1x_2^2 + x_1^2x_2^2) \, dx_2 \, dx_1 \]

\[ = \frac{4}{5} \int_0^1 \left( \frac{1}{2}x_1^2x_2^2 + \frac{1}{3}x_1x_2^3 + \frac{1}{3}x_1^2x_2^3 \right) \bigg|_0^1 \, dx_1 = \frac{4}{5} \int_0^1 \left( \frac{5}{6}x_1^2 + \frac{1}{3}x_1 \right) \, dx_1 \]
Covariance

Example. For the joint density example,

$$EX_1X_2 = \frac{4}{5} \int_0^1 \int_0^1 x_1 x_2 (x_1 + x_2 + x_1 x_2) \, dx_2 \, dx_1 = \frac{4}{5} \int_0^1 \int_0^1 (x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_2^2) \, dx_2 \, dx_1$$

$$= \frac{4}{5} \int_0^1 \left( \frac{1}{2} x_1^2 x_2^2 + \frac{1}{3} x_1 x_2^3 + \frac{1}{3} x_1^2 x_2^3 \right) \bigg|_1^1 \, dx_1 = \frac{4}{5} \int_0^1 \left( \frac{5}{6} x_1^2 + \frac{1}{3} x_1 \right) \, dx_1$$

$$= \frac{4}{5} \left( \frac{5}{18} x_1^3 + \frac{1}{6} x_1^2 \right) \bigg|_0^1$$

$$= \frac{16}{45}$$

Thus,

$$\text{Cov}(X_1, X_2) = \frac{16}{45} - \left( \frac{3}{5} \right)^2 = -\frac{1}{225}.$$
Covariance

Example. For the joint density example,

\[
EX_1X_2 = \frac{4}{5} \int_0^1 \int_0^1 x_1x_2(x_1 + x_2 + x_1x_2) \, dx_2 \, dx_1 = \frac{4}{5} \int_0^1 \int_0^1 (x_1^2x_2 + x_1x_2^2 + x_1^2x_2^2) \, dx_2 \, dx_1
\]

\[
= \frac{4}{5} \int_0^1 \left[ \frac{1}{2}x_1^2x_2^2 + \frac{1}{3}x_1x_2^3 + \frac{1}{3}x_1^2x_2^3 \right]_0^1 \, dx_1 = \frac{4}{5} \int_0^1 \left( \frac{5}{6}x_1^2 + \frac{1}{3}x_1 \right) \, dx_1
\]

\[
= \frac{4}{5} \left( \frac{5}{18}x_1^3 + \frac{1}{6}x_1^2 \right)_0^1 = \frac{4}{5} \left( \frac{5}{18} + \frac{1}{6} \right)
\]
Covariance

Example. For the joint density example,

\[
EX_1X_2 = \frac{4}{5} \int_0^1 \int_0^1 x_1 x_2 (x_1 + x_2 + x_1 x_2) \, dx_2 \, dx_1 = \frac{4}{5} \int_0^1 \int_0^1 (x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_2^2) \, dx_2 \, dx_1
\]

\[
= \frac{4}{5} \int_0^1 \left(\frac{1}{2} x_1^2 x_2 + \frac{1}{3} x_1 x_2^3 + \frac{1}{3} x_1^2 x_2^3\right) \bigg|_0^1 \, dx_1 = \frac{4}{5} \int_0^1 \left(\frac{5 x_1^2}{6} + \frac{1}{3} x_1\right) \, dx_1
\]

\[
= \frac{4}{5} \left(\frac{5}{18} x_1^3 + \frac{1}{6} x_1^2\right) \bigg|_0^1 = \frac{4}{5} \left(\frac{5}{18} + \frac{1}{6}\right) = \frac{16}{45}
\]
Covariance

Example. For the joint density example,

\[
EX_1X_2 = \frac{4}{5} \int_0^1 \int_0^1 x_1x_2(x_1 + x_2 + x_1x_2) \, dx_2 \, dx_1 = \frac{4}{5} \int_0^1 \int_0^1 (x_1^2x_2 + x_1x_2^2 + x_1^2x_2^2) \, dx_2 \, dx_1
\]

\[
= \frac{4}{5} \int_0^1 \left( \frac{1}{2}x_1^2x_2^2 + \frac{1}{3}x_1x_2^3 + \frac{1}{3}x_1^2x_2^3 \right) \bigg|_0^1 \, dx_1 = \frac{4}{5} \int_0^1 \left( \frac{5}{6}x_1^2 + \frac{1}{3}x_1 \right) \, dx_1
\]

\[
= \frac{4}{5} \left( \frac{5}{18}x_1^3 + \frac{1}{6}x_1^2 \right) \bigg|_0^1 = \frac{4}{5} \left( \frac{5}{18} + \frac{1}{6} \right) = \frac{16}{45}
\]

\[
EX_1 = EX_2
\]
Covariance

Example. For the joint density example,

\[ EX_1 X_2 = \frac{4}{5} \int_0^1 \int_0^1 x_1 x_2 (x_1 + x_2 + x_1 x_2) \, dx_2 \, dx_1 = \frac{4}{5} \int_0^1 \int_0^1 (x_1^2 x_2 + x_1 x_2^2 + x_1 x_2) \, dx_2 \, dx_1 \]

\[ = \frac{4}{5} \left[ \left( \frac{1}{2} x_1^2 x_2^2 + \frac{1}{3} x_1 x_2^3 + \frac{1}{3} x_1^2 x_2^3 \right) \right]_0^1 \, dx_1 = \frac{4}{5} \int_0^1 \left( \frac{5}{6} x_1^2 + \frac{1}{3} x_1 \right) \, dx_1 \]

\[ = \frac{4}{5} \left( \frac{5}{18} x_1^3 + \frac{1}{6} x_1^2 \right) \bigg|_0^1 = \frac{4}{5} \left( \frac{5}{18} + \frac{1}{6} \right) = \frac{16}{45} \]

\[ EX_1 = EX_2 = \frac{2}{5} \int_0^1 x_1 (3x_1 + 1) \, dx_1 = \]
Covariance

Example. For the joint density example,

\[
EX_1X_2 = \frac{4}{5} \int_0^1 \int_0^1 x_1x_2(x_1 + x_2 + x_1x_2) \, dx_2 \, dx_1 = \frac{4}{5} \int_0^1 \int_0^1 (x_1^2x_2 + x_1x_2^2 + x_1^2x_2^2) \, dx_2 \, dx_1
\]

\[
= \frac{4}{5} \int_0^1 \left( \frac{1}{2}x_1^2x_2^2 + \frac{1}{3}x_1x_2^3 + \frac{1}{3}x_1^2x_2^3 \right) \bigg|_0^1 \, dx_1 = \frac{4}{5} \int_0^1 \left( \frac{5}{6}x_1^2 + \frac{1}{3}x_1 \right) \, dx_1
\]

\[
= \frac{4}{5} \left( \frac{5}{18}x_1^3 + \frac{1}{6}x_1^2 \right) \bigg|_0^1 = \frac{4}{5} \left( \frac{5}{18} + \frac{1}{6} \right) = \frac{16}{45}
\]

\[
EX_1 = EX_2 = \frac{2}{5} \int_0^1 x_1(3x_1 + 1) \, dx_1 = \frac{2}{5} \left( x_1^3 + \frac{1}{2}x_1^2 \right) \bigg|_0^1
\]
Covariance

Example. For the joint density example,

\[ EX_1 X_2 = \frac{4}{5} \int_0^1 \int_0^1 x_1 x_2 (x_1 + x_2 + x_1 x_2) \, dx_2 \, dx_1 = \frac{4}{5} \int_0^1 \int_0^1 (x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_2^2) \, dx_2 \, dx_1 \]

\[ = \frac{4}{5} \int_0^1 \left( \frac{1}{2} x_1^2 x_2^2 + \frac{1}{3} x_1 x_2^3 + \frac{1}{3} x_1^2 x_2^3 \right) \bigg|_0^1 \, dx_1 = \frac{4}{5} \int_0^1 \left( \frac{5}{6} x_1^2 + \frac{1}{3} x_1 \right) \, dx_1 \]

\[ = \frac{4}{5} \left( \frac{5}{18} x_1^3 + \frac{1}{6} x_1^2 \right) \bigg|_0^1 = \frac{4}{5} \left( \frac{5}{18} + \frac{1}{6} \right) = \frac{16}{45} \]

\[ EX_1 = EX_2 = \frac{2}{5} \int_0^1 x_1 (3x_1 + 1) \, dx_1 = \frac{2}{5} \left( x_1^3 + \frac{1}{2} x_1^2 \right) \bigg|_0^1 = \frac{2}{5} \cdot \frac{3}{2} = \frac{3}{5} . \]

Thus,

\[ \text{Cov}(X_1, X_2) = \]
Covariance

Example. For the joint density example,

\[ EX_1X_2 = \frac{4}{5} \int_0^1 \int_0^1 x_1 x_2 (x_1 + x_2 + x_1 x_2) \, dx_2 \, dx_1 = \frac{4}{5} \int_0^1 \int_0^1 (x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_2^2) \, dx_2 \, dx_1 \]

\[ = \frac{4}{5} \int_0^1 \left( \frac{1}{2} x_1^2 x_2^2 + \frac{1}{3} x_1 x_2^3 + \frac{1}{3} x_1^2 x_2^3 \right) \bigg|_{x_1=0}^{x_1=1} \, dx_1 = \frac{4}{5} \int_0^1 \left( \frac{5}{6} x_1^2 + \frac{1}{3} x_1 \right) \, dx_1 \]

\[ = \frac{4}{5} \left( \frac{5}{18} x_1^3 + \frac{1}{6} x_1^2 \right) \bigg|_{x_1=0}^{x_1=1} = \frac{4}{5} \left( \frac{5}{18} \cdot \frac{1}{6} \right) = \frac{16}{45} \]

\[ EX_1 = EX_2 = \frac{2}{5} \int_0^1 x_1 (3x_1 + 1) \, dx_1 = \frac{2}{5} \left( x_1^3 + \frac{1}{2} x_1^2 \right) \bigg|_{x_1=0}^{x_1=1} = \frac{2}{5} \cdot \frac{3}{2} = \frac{3}{5} . \]

Thus,

\[ \text{Cov}(X_1, X_2) = \frac{16}{45} \]
**Covariance**

**Example.** For the joint density example,

\[
EX_1 X_2 = \frac{4}{5} \int_0^1 \int_0^1 x_1 x_2 (x_1 + x_2 + x_1 x_2) \, dx_2 \, dx_1 = \frac{4}{5} \int_0^1 \int_0^1 (x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_2^2) \, dx_2 \, dx_1
\]

\[
= \frac{4}{5} \int_0^1 \left( \frac{1}{2} x_1^2 x_2^2 + \frac{1}{3} x_1 x_2^3 + \frac{1}{3} x_1^2 x_2^3 \right) \bigg|_0^1 \, dx_1 = \frac{4}{5} \int_0^1 \left( \frac{5}{6} x_1^2 + \frac{1}{3} x_1 \right) \, dx_1
\]

\[
= \frac{4}{5} \left( \frac{5}{18} x_1^3 + \frac{1}{6} x_1^2 \right) \bigg|_0^1 = \frac{4}{5} \left( \frac{5}{18} + \frac{1}{6} \right) = \frac{16}{45}
\]

\[
EX_1 = EX_2 = \frac{2}{5} \int_0^1 x_1 (3x_1 + 1) \, dx_1 = \frac{2}{5} \left( x_1^3 + \frac{1}{2} x_1^2 \right) \bigg|_0^1 = \frac{2}{5} \cdot \frac{3}{2} = \frac{3}{5}.
\]

Thus,

\[
Cov(X_1, X_2) = \frac{16}{45} - \left( \frac{3}{5} \right)^2
\]
Covariance

Example. For the joint density example,

\[ EX_1 X_2 = \frac{4}{5} \int_0^1 \int_0^1 x_1 x_2 (x_1 + x_2 + x_1 x_2) \, dx_2 \, dx_1 = \frac{4}{5} \int_0^1 \int_0^1 (x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_2^2) \, dx_2 \, dx_1 \]

\[ = \frac{4}{5} \int_0^1 \left( \frac{1}{2} x_1^2 x_2 + \frac{1}{3} x_1 x_2^3 + \frac{1}{3} x_1^2 x_2^3 \right) \bigg|_0^1 \, dx_1 = \frac{4}{5} \int_0^1 \left( \frac{5}{6} x_1^2 + \frac{1}{3} x_1 \right) \, dx_1 \]

\[ = \frac{4}{5} \left( \frac{5}{18} x_1^3 + \frac{1}{6} x_1^2 \right) \bigg|_0^1 = \frac{4}{5} \left( \frac{5}{18} + \frac{1}{6} \right) = \frac{16}{45} \]

\[ EX_1 = EX_2 = \frac{2}{5} \int_0^1 x_1 (3x_1 + 1) \, dx_1 = \frac{2}{5} \left( x_1^3 + \frac{1}{2} x_1^2 \right) \bigg|_0^1 = \frac{2}{5} \cdot \frac{3}{2} = \frac{3}{5}. \]

Thus,

\[ \text{Cov}(X_1, X_2) = \frac{16}{45} - \left( \frac{3}{5} \right)^2 = \frac{80 - 81}{225} = -\frac{1}{225}. \]
Correlation

Covariance fails to take into account the scale of the measurements - larger values lead to larger covariance.

\[
\rho_{X_1, X_2} = \frac{\text{Cov}(X_1, X_2)}{\sigma_1 \sigma_2}
\]

In the example, \(\sigma_1^2 = \sigma_2^2 = 2.5\)

\[
\int_0^1 x^2 (3x + 1) \, dx - \left(\frac{3}{5}\right)^2 = 2.5 \cdot \frac{13}{12} - \frac{9}{25} = \frac{11}{150}
\]

and

\[
\rho_{X_1, X_2} = -\frac{1}{225} \cdot \frac{11}{150} = -\frac{2}{33} = -0.06
\]
Correlation

Covariance fails to take into account the scale of the measurements - larger values lead to larger covariance. Thus, we define the correlation to be the covariance of the standardized version of the random variables.

$$\rho_{X_1, X_2} = \frac{\text{Cov}(X_1, X_2)}{\sigma_1 \sigma_2}.$$
Correlation

Covariance fails to take into account the scale of the measurements - larger values lead to larger covariance. Thus, we define the correlation to be the covariance of the standardized version of the random variables.

\[ \rho_{X_1,X_2} = \frac{\text{Cov}(X_1, X_2)}{\sigma_1 \sigma_2} \]

In the example, \( \sigma_1 = \sigma_2 = 2.5 \):

\[ \int_{-1}^{1} x^2 (3x + 1) \, dx - \left( \frac{3}{5} \right)^2 = 2.5 \cdot \frac{13}{12} - \frac{9}{25} = \frac{11}{150} \]

and

\[ \rho_{X_1,X_2} = -\frac{1}{225} \cdot \frac{11}{150} = -\frac{2}{33} = -0.06 \]
Correlation

Covariance fails to take into account the scale of the measurements - larger values lead to larger covariance. Thus, we define the correlation to be the covariance of the standardized version of the random variables.

\[
\rho_{X_1, X_2} = \frac{\text{Cov}(X_1, X_2)}{\sigma_1 \sigma_2}.
\]
Correlation

Covariance fails to take into account the scale of the measurements - larger values lead to larger covariance. Thus, we define the correlation to be the covariance of the standardized version of the random variables.

\[
\rho_{X_1, X_2} = E \left[ \left( \frac{X_1 - \mu_1}{\sigma_1} \right) \left( \frac{X_2 - \mu_2}{\sigma_2} \right) \right] = \frac{\text{Cov}(X_1, X_2)}{\sigma_1 \sigma_2}.
\]

In the example,

\[
\sigma_1^2 = \sigma_2^2 = \frac{2}{5} \int_0^1 x^2(3x + 1) \, dx - \left( \frac{3}{5} \right)^2
\]
Correlation

Covariance fails to take into account the scale of the measurements - larger values lead to larger covariance. Thus, we define the **correlation** to be the covariance of the standardized version of the random variables.

$$
\rho_{X_1, X_2} = E \left[ \left( \frac{X_1 - \mu_1}{\sigma_1} \right) \left( \frac{X_2 - \mu_2}{\sigma_2} \right) \right] = \frac{\text{Cov}(X_1, X_2)}{\sigma_1 \sigma_2}.
$$

In the example,

$$
\sigma_1^2 = \sigma_2^2 = \frac{2}{5} \int_0^1 x^2(3x + 1) \, dx - \left( \frac{3}{5} \right)^2 = \frac{2}{5} \cdot \frac{13}{12} - \frac{9}{25} = \frac{11}{150}.
$$
Correlation

Covariance fails to take into account the scale of the measurements - larger values lead to larger covariance. Thus, we define the correlation to be the covariance of the standardized version of the random variables.

\[
\rho_{X_1, X_2} = \mathbb{E} \left[ \left( \frac{X_1 - \mu_1}{\sigma_1} \right) \left( \frac{X_2 - \mu_2}{\sigma_2} \right) \right] = \frac{\text{Cov}(X_1, X_2)}{\sigma_1 \sigma_2}.
\]

In the example,

\[
\sigma_1^2 = \sigma_2^2 = \frac{2}{5} \int_0^1 x^2(3x + 1) \, dx - \left( \frac{3}{5} \right)^2 = \frac{2}{5} \cdot \frac{13}{12} - \frac{9}{25} = \frac{11}{150}.
\]

and

\[
\rho_{X_1, X_2} = \frac{-1/225}{11/150} = -\frac{2}{33} = -0.06.
\]
Correlation

Covariance fails to take into account the scale of the measurements - larger values lead to larger covariance. Thus, we define the correlation to be the covariance of the standardized version of the random variables.

\[
\rho_{X_1, X_2} = \frac{\text{Cov}(X_1, X_2)}{\sigma_1 \sigma_2} = E \left[ \left( \frac{X_1 - \mu_1}{\sigma_1} \right) \left( \frac{X_2 - \mu_2}{\sigma_2} \right) \right] = \frac{\text{Cov}(X_1, X_2)}{\sigma_1 \sigma_2}.
\]

In the example,

\[
\sigma_1^2 = \sigma_2^2 = \frac{2}{5} \int_0^1 x^2(3x + 1) \, dx - \left( \frac{3}{5} \right)^2 = \frac{2}{5} \cdot \frac{13}{12} - \frac{9}{25} = \frac{11}{150}.
\]

and

\[
\rho_{X_1, X_2} = \frac{-1/225}{11/150} = -\frac{2}{33} = -0.06.
\]
Covariance

Exercise. \( \text{Cov}(a_1 + b_1 X_1, a_2 + b_2 X_2) = b_1 b_2 \text{Cov}(X_1, X_2) \)
Covariance

Exercise. Cov($a_1 + b_1 X_1$, $a_2 + b_2 X_2$) = $b_1 b_2 \text{Cov}(X_1, X_2)$

\[
\text{Cov}(a_1 + b_1 X_1, a_2 + b_2 X_2) = E[((a_1 + b_1 X_1) - (a_1 + b_1 \mu_1))((a_2 + b_2 X_2) - (a_2 + b_2 \mu_2))]
\]
Covariance

Exercise. $\text{Cov}(a_1 + b_1 X_1, a_2 + b_2 X_2) = b_1 b_2 \text{Cov}(X_1, X_2)$

\[
\text{Cov}(a_1 + b_1 X_1, a_2 + b_2 X_2) = E[((a_1 + b_1 X_1) - (a_1 + b_1 \mu_1))((a_2 + b_2 X_2) - (a_2 + b_2 \mu_2))] \\
= E[(b_1(X_1 - \mu_1))(b_2(X_2 - \mu_2))] \\
= b_1 b_2 \text{Cov}(X_1, X_2)
\]
Covariance

Exercise. \( \operatorname{Cov}(a_1 + b_1X_1, a_2 + b_2X_2) = b_1b_2\operatorname{Cov}(X_1, X_2) \)

\[
\operatorname{Cov}(a_1 + b_1X_1, a_2 + b_2X_2) = E[((a_1 + b_1X_1) - (a_1 + b_1\mu_1))(a_2 + b_2X_2) - (a_2 + b_2\mu_2))]
= E[(b_1(X_1 - \mu_1))(b_2(X_2 - \mu_2))]
= b_1b_2E[(X_1 - \mu_1)(X_2 - \mu_2)]
\]
Covariance

Exercise. $\text{Cov}(a_1 + b_1X_1, a_2 + b_2X_2) = b_1b_2\text{Cov}(X_1, X_2)$

\[
\text{Cov}(a_1 + b_1X_1, a_2 + b_2X_2) = E[((a_1 + b_1X_1) - (a_1 + b_1\mu_1))((a_2 + b_2X_2) - (a_2 + b_2\mu_2))] \\
= E[(b_1(X_1 - \mu_1))(b_2(X_2 - \mu_2))] \\
= b_1b_2E[(X_1 - \mu_1)(X_2 - \mu_2)] = b_1b_2\text{Cov}(X_1, X_2)
\]
Covariance

Exercise. \( \text{Cov}(a_1 + b_1 X_1, a_2 + b_2 X_2) = b_1 b_2 \text{Cov}(X_1, X_2) \)

\[
\text{Cov}(a_1 + b_1 X_1, a_2 + b_2 X_2) = E[((a_1 + b_1 X_1) - (a_1 + b_1 \mu_1))((a_2 + b_2 X_2) - (a_2 + b_2 \mu_2))] \\
= E[(b_1(X_1 - \mu_1))(b_2(X_2 - \mu_2))] \\
= b_1 b_2 E[(X_1 - \mu_1)(X_2 - \mu_2)] = b_1 b_2 \text{Cov}(X_1, X_2)
\]

Continuing, note that \( \text{Cov}(X_i, X_i) = \text{Var}(X_i) \)
Covariance

Exercise. $\text{Cov}(a_1 + b_1 X_1, a_2 + b_2 X_2) = b_1 b_2 \text{Cov}(X_1, X_2)$

\[
\text{Cov}(a_1 + b_1 X_1, a_2 + b_2 X_2) = E[((a_1 + b_1 X_1) - (a_1 + b_1 \mu_1))((a_2 + b_2 X_2) - (a_2 + b_2 \mu_2))] \\
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= b_1 b_2 E[(X_1 - \mu_1)(X_2 - \mu_2)] = b_1 b_2 \text{Cov}(X_1, X_2)
\]

Continuing, note that $\text{Cov}(X_i, X_i) = \text{Var}(X_i)$

\[
\text{Var}(b_1 X_1 + b_2 X_2) = E[((b_1 X_1 - b_1 \mu_1) + (b_2 X_2 - b_2 \mu_2))^2]
\]
Covariance

Exercise. \( \text{Cov}(a_1 + b_1 X_1, a_2 + b_2 X_2) = b_1 b_2 \text{Cov}(X_1, X_2) \)

\[
\text{Cov}(a_1 + b_1 X_1, a_2 + b_2 X_2) = E[((a_1 + b_1 X_1) - (a_1 + b_1 \mu_1))((a_2 + b_2 X_2) - (a_2 + b_2 \mu_2))] \\
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\]

Continuing, note that \( \text{Cov}(X_i, X_i) = \text{Var}(X_i) \)

\[
\text{Var}(b_1 X_1 + b_2 X_2) = E[((b_1 X_1 - b_1 \mu_1) + (b_2 X_2 - b_2 \mu_2))^2] \\
= E[((b_1 X_1 - b_1 \mu_1)^2] + 2E[(b_1 X_1 - b_1 \mu_1)(b_2 X_2 - b_2 \mu_2))] \\
+ E[(b_2 X_2 - b_2 \mu_2)^2]
\]
Covariance

Exercise. \( \text{Cov}(a_1 + b_1 X_1, a_2 + b_2 X_2) = b_1 b_2 \text{Cov}(X_1, X_2) \)

\[
\begin{align*}
\text{Cov}(a_1 + b_1 X_1, a_2 + b_2 X_2) & = E[((a_1 + b_1 X_1) - (a_1 + b_1 \mu_1))((a_2 + b_2 X_2) - (a_2 + b_2 \mu_2))] \\
& = E[(b_1 (X_1 - \mu_1))(b_2 (X_2 - \mu_2))] \\
& = b_1 b_2 E[(X_1 - \mu_1)(X_2 - \mu_2)] = b_1 b_2 \text{Cov}(X_1, X_2)
\end{align*}
\]

Continuing, note that \( \text{Cov}(X_i, X_i) = \text{Var}(X_i) \)

\[
\begin{align*}
\text{Var}(b_1 X_1 + b_2 X_2) & = E[((b_1 X_1 - b_1 \mu_1) + (b_2 X_2 - b_2 \mu_2))^2] \\
& = E[((b_1 X_1 - b_1 \mu_1)^2] + 2E[(b_1 X_1 - b_1 \mu_1)(b_2 X_2 - b_2 \mu_2))] \\
& \quad + E[(b_2 X_2 - b_2 \mu_2)^2] \\
& = b_1^2 \text{Var}(X_1) + 2b_1 b_2 \text{Cov}(X_1, X_2) + b_2^2 \text{Var}(X_2).
\end{align*}
\]
In particular,

\[ 0 \leq \sigma^2_{X_1 + cX_2} = \sigma_1^2 + 2c \rho_{X_1,X_2} \sigma_1 \sigma_2 + \sigma_2^2 c^2. \]
Covariance

In particular,

\[ 0 \leq \sigma_{X_1 + cX_2}^2 = \sigma_1^2 + 2c \rho_{X_1, X_2} \sigma_1 \sigma_2 + \sigma_2^2 c^2. \]

By considering the quadratic formula, we have that the discriminate
Covariance

In particular,

\[ 0 \leq \sigma_{X_1 + cX_2}^2 = \sigma_1^2 + 2c \rho_{X_1, X_2} \sigma_1 \sigma_2 + \sigma_2^2 c^2. \]

By considering the quadratic formula, we have that the discriminate

\[ 0 \geq (2 \rho_{X_1, X_2} \sigma_1 \sigma_2)^2 - 4 \sigma_1^2 \sigma_2^2 \]
Covariance

In particular,

\[ 0 \leq \sigma^2_{X_1 + cX_2} = \sigma^2_1 + 2c\rho_{X_1,X_2}\sigma_1\sigma_2 + \sigma^2_2c^2. \]

By considering the quadratic formula, we have that the discriminate

\[ 0 \geq (2\rho_{X_1,X_2}\sigma_1\sigma_2)^2 - 4\sigma^2_1\sigma^2_2 = (\rho^2_{X_1,X_2} - 1)4\sigma^2_1\sigma^2_2 \]
Covariance

In particular,

\[ 0 \leq \sigma_{X_1 + cX_2}^2 = \sigma_{X_1}^2 + 2c \rho_{X_1,X_2} \sigma_{X_1} \sigma_{X_2} + \sigma_{X_2}^2 c^2. \]

By considering the quadratic formula, we have that the discriminate

\[ 0 \geq (2 \rho_{X_1,X_2} \sigma_{X_1} \sigma_{X_2})^2 - 4 \sigma_{X_1}^2 \sigma_{X_2}^2 = (\rho_{X_1,X_2}^2 - 1)4 \sigma_{X_1}^2 \sigma_{X_2}^2 \quad \text{or} \quad \rho_{X_1,X_2}^2 \leq 1. \]
In particular,

\[ 0 \leq \sigma_{X_1+cX_2}^2 = \sigma_1^2 + 2c \rho_{X_1,X_2} \sigma_1 \sigma_2 + \sigma_2^2 c^2. \]

By considering the quadratic formula, we have that the discriminate

\[ 0 \geq (2 \rho_{X_1,X_2} \sigma_1 \sigma_2)^2 - 4 \sigma_1^2 \sigma_2^2 = (\rho_{X_1,X_2}^2 - 1)4 \sigma_1^2 \sigma_2^2 \quad \text{or} \quad \rho_{X_1,X_2}^2 \leq 1. \]

Consequently, \(-1 \leq \rho_{X_1,X_2} \leq 1\).

When we have \(|\rho_{X_1,X_2}| = 1\), we also have for some value of \(c\) that

\[ \sigma_{X_1+cX_2}^2 = 0. \]
Covariance

In particular,

\[ 0 \leq \sigma^2_{X_1 + cX_2} = \sigma^2_1 + 2c \rho_{X_1,X_2} \sigma_1 \sigma_2 + \sigma^2_2 c^2. \]

By considering the quadratic formula, we have that the discriminate

\[ 0 \geq (2 \rho_{X_1,X_2} \sigma_1 \sigma_2)^2 - 4 \sigma^2_1 \sigma^2_2 = (\rho^2_{X_1,X_2} - 1)4 \sigma^2_1 \sigma^2_2 \quad \text{or} \quad \rho^2_{X_1,X_2} \leq 1. \]

Consequently, \(-1 \leq \rho_{X_1,X_2} \leq 1\).

When we have \(|\rho_{X_1,X_2}| = 1\), we also have for some value of \(c\) that

\[ \sigma^2_{X_1 + cX_2} = 0. \]

In this case, \(X_1 + cX_2\) is a constant random variable.
Covariance

In particular,

$$0 \leq \sigma_{X_1+cX_2}^2 = \sigma_1^2 + 2c \rho_{X_1,X_2} \sigma_1 \sigma_2 + \sigma_2^2 c^2.$$ 

By considering the quadratic formula, we have that the discriminate

$$0 \geq (2 \rho_{X_1,X_2} \sigma_1 \sigma_2)^2 - 4 \sigma_1^2 \sigma_2^2 \rho_{X_1,X_2}^2 = (\rho_{X_1,X_2}^2 - 1)4 \sigma_1^2 \sigma_2^2 \text{ or } \rho_{X_1,X_2}^2 \leq 1.$$ 

Consequently, $$-1 \leq \rho_{X_1,X_2} \leq 1.$$ 

When we have $$|\rho_{X_1,X_2}| = 1$$, we also have for some value of $$c$$ that

$$\sigma_{X_1+cX_2}^2 = 0.$$ 

In this case, $$X_1 + cX_2$$ is a constant random variable and $$X_1$$ and $$X_2$$ are linearly related.
Covariance

In particular,

$$0 \leq \sigma_{X_1 + cX_2}^2 = \sigma_1^2 + 2c \rho_{X_1, X_2} \sigma_1 \sigma_2 + \sigma_2^2 c^2.$$ 

By considering the quadratic formula, we have that the discriminate

$$0 \geq (2 \rho_{X_1, X_2} \sigma_1 \sigma_2)^2 - 4 \sigma_1^2 \sigma_2^2 = (\rho_{X_1, X_2}^2 - 1)4 \sigma_1^2 \sigma_2^2 \quad \text{or} \quad \rho_{X_1, X_2}^2 \leq 1.$$ 

Consequently, $$-1 \leq \rho_{X_1, X_2} \leq 1.$$ 

When we have $$|\rho_{X_1, X_2}| = 1,$$ we also have for some value of $$c$$ that

$$\sigma_{X_1 + cX_2}^2 = 0.$$ 

In this case, $$X_1 + cX_2$$ is a constant random variable and $$X_1$$ and $$X_2$$ are linearly related. In this case, the sign of $$\rho_{X_1, X_2}$$ depends on the sign of the linear relationship.
Covariance

For the case \( c = 1 \), the variance \( \sigma_{X_1+X_2} \),

\[
\sigma_{X_1+X_2}^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + 2\rho\sigma_{X_1}\sigma_{X_2}.
\]

Notice the analogy between this formula and the law of cosines:

\[
c^2 = a^2 + b^2 - 2ab\cos\theta.
\]

If the two observations are uncorrelated, we have the Pythagorean identity

\[
\sigma_{X_1+X_2}^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2.
\]
Covariance

For the case $c = 1$, the variance $\sigma_{X_1+X_2}$, we have

$$\sigma_{X_1+X_2}^2 = \sigma_1^2 + \sigma_2^2 + 2\rho \sigma_1 \sigma_2.$$
Covariance

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Figure: For the law of cosines, let $a = \sigma_{X_1}$, $b = \sigma_{X_2}$, $\sigma_{X_1 + X_2}$ and $r = -\cos\theta$
Covariance

For the case $c = 1$, the variance $\sigma_{X_1 + X_2}$, we have

$$\sigma_{X_1 + X_2}^2 = \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2.$$  

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Figure: For the law of cosines, let $a = \sigma_{X_1}$, $b = \sigma_{X_2}$, $\sigma_{X_1 + X_2}$ and $r = -\cos\theta$. 


Covariance

For the case $c = 1$, the variance $\sigma_{X_1 + X_2}$, we have

$$\sigma^2_{X_1 + X_2} = \sigma^2_1 + \sigma^2_2 + 2 \rho \sigma_1 \sigma_2.$$  

Notice the analogy between this formula and the law of cosines: $c^2 = a^2 + b^2 - 2ab \cos \theta$.

If the two observations are uncorrelated, we have the Pythagorean identity $\sigma^2_{X_1 + X_2} = \sigma^2_{X_1} + \sigma^2_{X_2}$.

**Figure:** For the law of cosines, let $a = \sigma_{X_1}, b = \sigma_{X_2}, \sigma_{X_1 + X_2}$ and $r = - \cos \theta$
Covariance

For the case $c = 1$, the variance $\sigma_{X_1+X_2}$, we have

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More generally, for $X_i$, $i = 1, \ldots, n$,

Figure: For the law of cosines, let $a = \sigma_{X_1}, b = \sigma_{X_2}, \sigma_{X_1+X_2}$ and $r = -\cos \theta$
Covariance

For the case $c = 1$, the variance $\sigma_{X_1+X_2}$, we have

$$\sigma_{X_1+X_2}^2 = \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2.$$  

Notice the analogy between this formula and the law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\theta$.

If the two observations are uncorrelated, we have the Pythagorean identity $\sigma_{X_1+X_2}^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2$.

More generally, for $X_i, i = 1, \ldots, n$,

$$\text{Var} \left( \sum_{i=1}^{n} b_i X_i \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_i b_j \text{Cov}(X_i, X_j)$$

Figure: For the law of cosines, let $a = \sigma_{X_1}, b = \sigma_{X_2}, \sigma_{X_1+X_2}$ and $r = -\cos\theta$.
Hypergeometric Random Variables

Consider an urn with $m$ white balls and $n$ black balls. Remove $k$
Hypergeometric Random Variables

Consider an urn with \( m \) white balls and \( n \) black balls. Remove \( k \) and set

\[
X_i = \begin{cases} 
0 & \text{if the } i\text{-th ball is black,} \\
\end{cases}
\]

Thus, 

\[
E[X_i] = \frac{m}{m+n} \quad \text{and} \quad \text{Var}(X_i) = \frac{mn}{(m+n)^2}
\]

For \( i \neq j \), 

\[
E[X_i X_j] = P\{X_i = 1, X_j = 1\} = P\{X_i = 1 | X_j = 1\} P\{X_j = 1\} = \frac{m-1}{m+n-1} \cdot \frac{m}{m+n}
\]

\[
\text{Cov}(X_i, X_j) = \frac{mn}{(m+n)^2} - \left(\frac{m}{m+n}\right)^2 = \frac{mn}{(m+n)^2} \left(1 - \frac{m}{m+n}\right)
\]
Hypergeometric Random Variables

Consider an urn with $m$ white balls and $n$ black balls. Remove $k$ and set

$$X_i = \begin{cases} 
0 & \text{if the } i\text{-th ball is black,} \\
1 & \text{if the } i\text{-th ball is white.}
\end{cases}$$

Thus, $E[X_i] = \frac{m}{m+n}$ and $\text{Var}(X_i) = \frac{mn}{(m+n)^2}$. For $i \neq j$,

$$E[X_i X_j] = P\{X_i = 1, X_j = 1\} = P\{X_i = 1|X_j = 1\} P\{X_j = 1\} = \frac{m-1}{m+n-1} \cdot \frac{m}{m+n}.$$
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\end{cases}$$

$$X_i \sim Ber\left(\frac{m}{m+n}\right).$$
Hypergeometric Random Variables

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$X_i \sim Ber\left(\frac{m}{m+n}\right)$. Thus, $EX_i = \frac{m}{m+n}$ and $\text{Var}(X_i) = \frac{mn}{(m+n)^2}$. 
Hypergeometric Random Variables

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\end{cases}$$

$x_i \sim Ber\left(\frac{m}{m+n}\right)$. Thus, $EX_i = \frac{m}{m+n}$ and $Var(X_i) = \frac{mn}{(m+n)^2}$. For $i \neq j$,

$$E[X_iX_j] = P\{X_i = 1, X_j = 1\}$$
Hypergeometric Random Variables

Consider an urn with \( m \) white balls and \( n \) black balls. Remove \( k \) and set

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X_i = \begin{cases} 
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\( X_i \sim Ber\left( \frac{m}{m+n} \right) \). Thus, \( E X_i = \frac{m}{m+n} \) and \( \text{Var}(X_i) = \frac{mn}{(m+n)^2} \). For \( i \neq j \),

\[
E[X_i X_j] = P\{X_i = 1, X_j = 1\} = P\{X_i = 1|X_j = 1\} P\{X_j = 1\}
\]
Hypergeometric Random Variables

Consider an urn with $m$ white balls and $n$ black balls. Remove $k$ and set

$$X_i = \begin{cases} 
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\end{cases}$$

$X_i \sim \text{Ber}\left(\frac{m}{m+n}\right)$. Thus, $EX_i = \frac{m}{m+n}$ and $\text{Var}(X_i) = \frac{mn}{(m+n)^2}$. For $i \neq j$,

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\end{cases}
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\( X_i \sim \text{Ber} \left( \frac{m}{m+n} \right) \). Thus, \( EX_i = \frac{m}{m+n} \) and \( \text{Var}(X_i) = \frac{mn}{(m+n)^2} \). For \( i \neq j \),

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\]

\[
\text{Cov}(X_i, X_j) = \frac{m(m-1)}{(m+n)(m+n-1)}
\]
Hypergeometric Random Variables

Consider an urn with $m$ white balls and $n$ black balls. Remove $k$ and set

$$X_i = \begin{cases} 
0 & \text{if the } i\text{-th ball is black,} \\
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$$E[X_iX_j] = P\{X_i = 1, X_j = 1\} = P\{X_i = 1|X_j = 1\}P\{X_j = 1\} = \frac{m-1}{m+n-1} \cdot \frac{m}{m+n}.$$

$$\text{Cov}(X_i, X_j) = \frac{m(m-1)}{(m+n)(m+n-1)} - \left(\frac{m}{m+n}\right)^2$$
Hypergeometric Random Variables

Consider an urn with \( m \) white balls and \( n \) black balls. Remove \( k \) and set

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\]

\[
\text{Cov}(X_i, X_j) = \frac{m(m-1)}{(m+n)(m+n-1)} - \left(\frac{m}{m+n}\right)^2 = \frac{m}{m+n} \left(\frac{m-1}{m+n-1} - \frac{m}{m+n}\right)
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Hypergeometric Random Variables

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$X_i \sim \text{Ber}(\frac{m}{m+n})$. Thus, $EX_i = \frac{m}{m+n}$ and $\text{Var}(X_i) = \frac{mn}{(m+n)^2}$. For $i \neq j$,

$$E[X_iX_j] = P\{X_i = 1, X_j = 1\} = P\{X_i = 1|X_j = 1\}P\{X_j = 1\} = \frac{m-1}{m+n-1} \cdot \frac{m}{m+n}.$$

$$\text{Cov}(X_i, X_j) = \frac{m(m-1)}{(m+n)(m+n-1)} - \left(\frac{m}{m+n}\right)^2 = \frac{m}{m+n} \left(\frac{m-1}{m+n-1} - \frac{m}{m+n}\right)$$

$$= \frac{m}{m+n} \left(\frac{-n}{(m+n)(m+n-1)}\right)$$
Hypergeometric Random Variables

Consider an urn with $m$ white balls and $n$ black balls. Remove $k$ and set

$$X_i = \begin{cases} 
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1 & \text{if the } i\text{-th ball is white.} 
\end{cases}$$

$X_i \sim Ber\left(\frac{m}{m+n}\right)$. Thus, $EX_i = \frac{m}{m+n}$ and $Var(X_i) = \frac{mn}{(m+n)^2}$. For $i \neq j$,

$$E[X_iX_j] = P\{X_i = 1, X_j = 1\} = P\{X_i = 1|X_j = 1\}P\{X_j = 1\} = \frac{m-1}{m+n-1} \cdot \frac{m}{m+n}.$$

$$Cov(X_i, X_j) = \frac{m(m-1)}{(m+n)(m+n-1)} - \left(\frac{m}{m+n}\right)^2 = \frac{m}{m+n} \left(\frac{m-1}{m+n-1} - \frac{m}{m+n}\right)$$

$$= \frac{m}{m+n} \left(\frac{-n}{(m+n)(m+n-1)}\right) = \frac{-mn}{(m+n)^2(m+n-1)}$$
Hypergeometric Random Variables

The correlation

\[ \rho_{X_i, X_j} \]
The correlation

$$\rho_{X_i, X_j} = \frac{\text{Cov}(X_i, X_j)}{\text{Var}(X_i)}$$
Hypergeometric Random Variables

The correlation

$$\rho_{X_i, X_j} = \frac{\text{Cov}(X_i, X_j)}{\text{Var}(X_i)} = \frac{-mn}{(m+n)^2(m+n-1)}$$
Hypergeometric Random Variables

The correlation

$$\rho_{X_i, X_j} = \frac{\text{Cov}(X_i, X_j)}{\text{Var}(X_i)} = \frac{-mn}{(m + n)^2(m + n - 1)} \div \frac{mn}{(m + n)^2} = -\frac{1}{m + n - 1}$$
Hypergeometric Random Variables

The correlation

$$\rho_{X_i, X_j} = \frac{\text{Cov}(X_i, X_j)}{\text{Var}(X_i)} = \frac{-mn}{(m+n)^2(m+n-1)} / \frac{mn}{(m+n)^2} = -\frac{1}{m+n-1}$$

Let $X = X_1 + X_2 + \cdots + X_k$ denote the number of white balls. Then,

$$\text{Var}(X) = \sum_{i=1}^{k} \sum_{j=1}^{k} \text{Cov}(X_i, X_j)$$
Hypergeometric Random Variables

The correlation

\[ \rho_{X_i, X_j} = \frac{\text{Cov}(X_i, X_j)}{\text{Var}(X_i)} = \frac{-mn}{(m+n)^2(m+n-1)} \cdot \frac{mn}{(m+n)^2} = -\frac{1}{m+n-1} \]

Let \( X = X_1 + X_2 + \cdots + X_k \) denote the number of white balls. Then,

\[ \text{Var}(X) = \sum_{i=1}^{k} \sum_{j=1}^{k} \text{Cov}(X_i, X_j) = \sum_{i=1}^{k} \text{Var}(X_i) \]
Hypergeometric Random Variables

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$$\rho_{X_i, X_j} = \frac{\text{Cov}(X_i, X_j)}{\text{Var}(X_i)} = \frac{-mn}{(m+n)^2(m+n-1)} \div \frac{mn}{(m+n)^2} = -\frac{1}{m+n-1}$$

Let $X = X_1 + X_2 + \cdots + X_k$ denote the number of white balls. Then,

$$\text{Var}(X) = \sum_{i=1}^{k} \sum_{j=1}^{k} \text{Cov}(X_i, X_j) = \sum_{i=1}^{k} \text{Var}(X_i) + \sum_{i=1}^{k} \sum_{j \neq i} \text{Cov}(X_i, X_j)$$
Hypergeometric Random Variables

The correlation

$$\rho_{x_i,x_j} = \frac{\text{Cov}(X_i, X_j)}{\text{Var}(X_i)} = \frac{-mn}{(m+n)^2(m+n-1)} \left/ \frac{mn}{(m+n)^2} \right. = -\frac{1}{m+n-1}$$

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$$= k \frac{mn}{(m+n)^2}$$
Hypergeometric Random Variables

The correlation

\[ \rho_{X_i, X_j} = \frac{\text{Cov}(X_i, X_j)}{\text{Var}(X_i)} = \frac{-mn}{(m+n)^2(m+n-1)} \times \frac{mn}{(m+n)^2} = -\frac{1}{m+n-1} \]

Let \( X = X_1 + X_2 + \cdots + X_k \) denote the number of white balls. Then,

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\]

\[
= k \frac{mn}{(m+n)^2} + k(k-1) \left( \frac{-mn}{(m+n)^2(m+n-1)} \right)
\]
Hypergeometric Random Variables

The correlation
\[
\rho_{X_i, X_j} = \frac{\text{Cov}(X_i, X_j)}{\text{Var}(X_i)} = \frac{-mn}{(m+n)^2(m+n-1)} / \frac{mn}{(m+n)^2} = -\frac{1}{m+n-1}
\]

Let \( X = X_1 + X_2 + \cdots + X_k \) denote the number of white balls. Then,

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\]

\[
= k \frac{mn}{(m+n)^2} \left( 1 - \frac{k-1}{m+n-1} \right) = kp(1-p) \frac{N-k}{N-1}
\]
The correlation \( \rho_{X_i,X_j} = \frac{\text{Cov}(X_i, X_j)}{\text{Var}(X_i)} = \frac{-mn}{(m+n)^2(m+n-1)} \frac{mn}{(m+n)^2} = -\frac{1}{m+n-1} \)

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where \( N = m + n \) is the total number of balls.
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The correlation

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Let \( X = X_1 + X_2 + \cdots + X_k \) denote the number of white balls. Then,

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= k \frac{mn}{(m+n)^2} \left( 1 - \frac{k-1}{m+n-1} \right) = kp(1-p) \frac{N-k}{N-1}
\]

where \( N = m + n \) is the total number of balls and \( p = m/(m+n) \) is the probability that a white ball is chosen.