1 Form and Classification

We will move on the second order linear ordinary differential equations

\[ a(x)y'' + b(x)y' + c(x)y = 0 \]  \hspace{1cm} (1)

where \( a, b, \) and \( c \) are analytical functions.

The standard form (1) is

\[ y'' + p(x)y' + q(x)y = 0 \]  \hspace{1cm} (2)

with

\[ p(x) = \frac{b(x)}{a(x)} \quad \text{and} \quad q(x) = \frac{c(x)}{a(x)}. \]

A point \( x_0 \) is called an ordinary point of equation (2) if both \( p \) and \( q \) are analytic at \( x_0 \). If \( x_0 \) is not an ordinary point, it is called a singular point.

If \( a, b \) and \( c \) are analytic, then \( x_0 \) is a regular point as long as \( a(x_0) \neq 0 \). We can also include \( x_0 \) as an ordinary point if it has a removable singularity. For example, if \( a(x) = x \) and \( c(x) = \sin x \), then

\[ q(x) = \frac{\sin x}{x}. \]

So the singular can be removed by defining at \( x_0 = 0 \)

\[ q(0) = \lim_{x \to 0} q(x) = 1. \]

and \( q \) is analytic at 0.

By examining 2), we can see continue taking derivatives, giving equations in higher and higher order derivatives in \( y \) as function of the lower order derivatives of \( y \) and multiple derivatives of \( p \) and \( q \). Thus, we will use techniques based on the fact that the solution

\[ y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n. \]

can be realized as a power series solution about an ordinary point \( x_0 \).
2 First Order Example

For the first order differential equation, 
\[ y' + 2xy = 0, \]
the integrating factor is \( \exp(x^2) \). So the solution 
\[ y(x) = A \exp(-x^2) = A \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} = A \left( 1 - x^2 + \frac{1}{2} x^4 - \frac{1}{6} x^6 + \frac{1}{24} x^8 - \cdots \right). \]

Now, let’s look to develop the techniques that will lead to a series solution at \( x_0 = 0 \).

\[ y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \cdots. \]
\[ y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \cdots. \]
\[ 2xy(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = 2a_0 x + 2a_1 x^2 + 2a_2 x^3 + 2a_3 x^4 \cdots. \]

To make the summation more transparent, we shift the indexing on the sums so that the powers on \( x \) match.

\[ y'(x) = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \cdots. \]
\[ 2xy(x) = \sum_{n=1}^{\infty} a_{n-1} x^n = 0 + 2a_0 x + 2a_1 x^2 + 2a_2 x^3 + 2a_3 x^4 \cdots. \]

By the uniqueness of power series, each of the coefficients of \( x_n \) on the right side must equal 0. Thus,
\[ a_1 = 0, \quad 2a_2 + 2a_0 = 0, \quad 3a_3 + 2a_1 = 0, \quad 4a_4 + 2a_2 = 0, \quad \cdots (n + 1)a_{n+1} + 2a_{n-1} = 0. \]

We have written the first few instances of the **recursion relation**
\[ (n + 1)a_{n+1} + 2a_{n-1} = 0 \]
or again by shift the indices
\[ (n + 2)a_{n+2} + 2a_n = 0, \quad a_{n+2} = -\frac{2}{n+2} a_n. \]

Using the fact that \( a_1 = 0 \) and the sequence of equations for the coefficients, we see that \( a_n = 0 \) for all odd
values of $n$. For the even values of $n$

\[
a_2 = -\frac{2}{3}a_0 = -a_0
\]

\[
a_4 = -\frac{2}{5}a_2 = -\frac{1}{2}a_2
\]

\[
a_6 = -\frac{2}{7}a_4 = -\frac{1}{3}a_4
\]

\[
a_8 = -\frac{2}{9}a_6 = -\frac{1}{4}a_6
\]

\[\vdots = \vdots = \vdots = \vdots\]

\[
a_{2k} = -\frac{2}{2k}a_{2(k-1)} = -\frac{1}{k}a_{2(k-1)} = (-1)^k \frac{1}{k-\frac{3}{2}}a_0 = (-1)^k \frac{1}{k!}a_0
\]

3 Bessel Functions

Bessel functions were first defined by the mathematician Daniel Bernoulli and then generalized by Friedrich Bessel, as solutions

\[
x^2 y'' + xy' + (x^2 - \alpha^2)y = 0
\]

the parameter $\alpha$ is called the the order of the Bessel equation and function.

\[
y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \cdots
\]

\[
y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \cdots
\]

\[
y''(x) = \sum_{n=0}^{\infty} (n(n-1)) a_n x^{n-2} = 2 \cdot 1a_2 + 2 \cdot 2a_3 x + 3 \cdot 3a_4 x^2 + 4 \cdot 4a_5 x^3 \cdots
\]

Let’s look for a power series solution for the Bessel equation of order $\alpha = 0$.

\[
x^2 y(x) = \sum_{n=0}^{\infty} a_n x^{n+2} = a_0 x^2 + a_1 x^3 + a_2 x^4 + \cdots
\]

\[
x y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4 \cdots
\]

\[
x^2 y''(x) = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = 2 \cdot 1a_2 x^2 + 3 \cdot 2a_3 x^3 + 4 \cdot 3a_4 x^4 + \cdots
\]

Again, we shift the indexing on the sums so that the powers on $x$ match.

\[
x^2 y(x) = \sum_{n=2}^{\infty} a_{n-2} x^n = a_0 x^2 + a_1 x^3 + a_2 x^4 + \cdots
\]

\[
x y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4 \cdots
\]

\[
x^2 y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = 2 \cdot 1a_2 x^2 + 3 \cdot 2a_3 x^3 + 4 \cdot 3a_4 x^4 + \cdots
\]
\[ 0 = x^2 y''(x) + xy'(x) + x^2 y(0) \]
\[ = a_1 x + \sum_{n=2}^{\infty} (a_{n-2} + na_n + n(n-1)a_{n-2}) x^n \]
\[ = a_1 x + \sum_{n=2}^{\infty} (a_{n-2} + n^2 a_n) x^n \]
\[ = a_1 x + (a_0) + (a_0 + 2^2 a_2) x^2 + (a_1 + 3^2 a_3) x^3 + \cdots \]

Thus,
\[ a_1 = 0, \quad a_0 + 2^2 a_2 = 0, \quad a_1 + 3^2 a_3 = 0, \quad a_2 + 4^2 a_4 = 0, \quad \cdots \quad a_{n-2} + n^2 a_n = 0. \]

Again, the recursion relations ensures that \( a_n = 0 \) for odd values of \( n \). For the even values of \( n \)
\[ a_2 = -\frac{1}{2^2} a_0 = -\frac{1}{2^2} a_0 \]
\[ a_4 = -\frac{1}{4^2} a_2 = \frac{1}{4^2} a_0 = \frac{1}{4^2} a_0 \]
\[ a_6 = -\frac{1}{6^2} a_2 = \frac{1}{6^2} a_0 = \frac{1}{6^2} a_0 \]
\[ a_8 = -\frac{2}{8^2} a_6 = \frac{1}{8^2} a_0 = -\frac{1}{8^2} a_0 \]
\[ \vdots \]
\[ -a_{2k} = -\frac{1}{k^2} a_{2(k-1)} = (-1)^k \frac{1}{2^k (k!)^2} a_0 \]

In writing (3) in the standard form (2)
\[ y'' + \frac{1}{x} y' + \frac{x^2 - \alpha^2}{x^2} y = 0, \]
we see that \( x = 0 \) is a singular point. This will lead to the fact that (3) will have solutions that are unbounded at \( x = 0 \). Bessel functions of the first kind, denoted as \( J_\alpha(x) \), are solutions of that are finite at the origin \( x = 0 \) for integer or positive \( \alpha \). The usual form for \( J_0(x) \), has \( J_0(0) = 1 \) and \( J'_0(0) = 0. \) Thus \( a_0 = 1 \) and \( a_1 = 0. \)

We can write the solution as
\[ J_0(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2^k (k!)^2} x^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left( \frac{x}{2} \right)^{2k}. \]

### 4 Equations with Analytic Coefficients

For \( p \) and \( q \) analytic function in equation (1) and \( x_0 \) an ordinary point for this equation. Then (1) has two linearly independent analytic solutions of the form
\[ y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n. \]

The radius of convergence of any power series solution is at least as large as the distance from \( x_0 \) to the nearest singular point (real or complex-valued).
Exercise 1. Give the ensured radius of convergence about the ordinary point $x_0 = 0$

- $y'' + 4x^2 y + y = 0$
- $(1 + x^2)y'' + 6xy + (\sin x)y = 0$
- $(1 + x)y'' + (1 - x^2)y + xy = 0$

We can carry out the procedure for any case in which $p$ and $q$ are analytic. The process can be involve if either $p$ or $q$ have a power series solutions with infinitely many terms. For example, take

$$y'' + e^x y' + (1 + x)y = 0,$$

then

$$(1 + x)y(x) = (1 + x) \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} (a_{n-1} + a_n) x^n.$$

$$e^x y'(x) = (\sum_{n=1}^{\infty} n a_n x^n) \left( \sum_{n=0}^{\infty} \frac{1}{n!} x^n \right).$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

The process is in principle the same - give the Cauchy product to $e^x y'(x)$, shift the summation indices, if necessary, set the coefficients of $x^n$ to zero and develop recursion relations.