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- The domain of the functions involved (usually denoted Ω) is an essential detail.
- Often other conditions placed on domain boundary, denoted $\partial\Omega$.

History

Jean-Baptiste le Rond d'Alembert

Joseph Fourier

Pierre-Simon Laplace

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- Co-evolved with physical sciences: mechanics, electromagnetism, quantum mechanics
- Modern uses are everywhere: chemistry, ecology, finance, image processing, big data

Some famous examples

Name	Equation	Application
Wave equation	$u_{tt} = c^2 u_{xx}$	Vibrating string
Diffusion equation	$u_t = D u_{xx}$	Heat flow
Laplace's equation	$u_{xx} + u_{yy} = 0$	Electrostatics etc.
Burger's equation	$u_t + u u_x = D u_{xx}$	Fluid mechanics
Cahn-Hilliard equation	$u_t = (u^3 - u - u_{xx})_{xx}$	Phase separation
Eikonal equation	$\lvert \nabla u \rvert = f(x)$	Optics
Fisher's equation	$u_t = u_{xx} + u(1-u)$	Ecology
Korteweg-de Vries eqn.	$u_t + 6 u u_x + u_{xxx} = 0$	Water waves
Schrödinger eqn.	$i u_t + u_{xx} = 0$	Quantum mech.
Nonlinear Schrödinger eqn.	$i u_t + u_{xx} + \lvert u \rvert^2 u = 0$	Nonlinear optics
Swift-Hohenberg eqn.	$u_t = -(\partial_{xx} + 1)^2 u + N(u)$	Pattern formation

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- In both cases, if $f \equiv 0$ BCs are called *homogeneous*.
- *Periodic*: assumes function is periodic, smooth in one or more variables, for example when using polar coordinates:

$$\lim_{\theta \to 0^+} u(r, \theta) = \lim_{\theta \to 2\pi^-} u(r, \theta)$$

plus same for derivatives.

Side conditions,cont.

Other common side conditions:

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- If time is an independent variable, there can be initial conditions as with ODEs, e.g. requiring solution $u(x, t)$ to satisfy $u(x, 0) = g(x)$.

Well-posedness

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- Uniqueness: there is only one solution.
- Stability: if the model changes a little, then the solution only changes a little.

Notation of derivatives and integrals

- Partial derivative notation

$$\frac{\partial}{\partial x}u(x,y) = \left.\frac{d}{dx}\right|_y u(x,y) = \partial_x u = u_x.$$

and multiple derivatives $\partial_x \partial_y u = u_{xy}$.

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- To actually compute integrals, sometimes go to coordinates and write as an iterated integral.

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  Try $u = f(x + bt)$ where b, f unknown. Plug in:

  $$bf'(x + bt) + cf'(x + bt) = 0.$$

  To make this true, we need $b = -c$, so $u = f(x - ct)$ for any smooth f.