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or

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Plugging in gives dispersion relation $\omega = \omega(k)$ or $\sigma = \sigma(k)$.

Examples

For usual wave equation

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For the diffusion equation

$$u_t = Du_{xx},$$

same process gives $\sigma(k) = -Dk^2$, i.e. solutions decay of $k \neq zero$.

Phase and group velocity of waves

For a real dispersion relation $\omega(k)$, there are solutions

$$u(x,t) = \exp\Big(ikx - i\omega(k)t\Big) = \exp\left(ik\left[x - \frac{\omega(k)}{k}t\right]\right),$$

which are waves traveling at speed $\omega(k)/k$. This is the *phase velocity*. If the phase velocities ω/k are different, equation is called *dispersive*.

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Integral is a traveling wave moving at speed $\omega'(k_0)$. This is known as the *group velocity*.

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Animation of phase and group velocity

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Suppose a linear equation has solutions $u(x, t) = \exp(\sigma t + ikx)$ where $\sigma = \sigma(k)$ is the (real exponential form) dispersion relation.

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Inserting $u = \exp(\sigma t + ikx)$ gives $\sigma = -k^2 + iAk + B$.
For $B < 0$, Re $\sigma < 0$, therefore linearly stable.
For $B > 0$, Re $\sigma > 0$ for small k, therefore linearly unstable.
For $B = 0$, marginally stable since Re $\sigma(0) = 0$.

Steady state solutions

Consider generic linear or nonlinear PDE of form

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Remarks:

- u_0 solves an ODE
- u_0 is usually subject to boundary/ far field conditions
- If $u(x, 0) = u_0(x)$, then $u(x, t) = u_0(x)$ for all $t > 0$.
- Can be many solutions, esp. for nonlinear equations

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Solution is easy: $u_0 = x$.

Steady state solutions, example 2

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Trick to solving: multiply by u_x and integrate.

$$\int u_{xx}u_x + 2u(1-u^2)u_x \, dx = \frac{1}{2}u_x^2 + u^2 - \frac{1}{2}u^4 + C = 0,$$

which uses $u_{xx}u_x = \frac{1}{2}(u_x^2)_x$ and $f'(u)u_x = f(u)_x$.

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Since $u(\pm\infty) = \pm 1$, $C = -1/2$.

Steady state solutions, example 3, cont.

First order equation can now be written

$$u_x = \sqrt{u^4 - 2u^2 + 1} = 1 - u^2,$$

Steady state solutions, example 3, cont.

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which can be solved by separating variables

$$\frac{du}{1-u^2} = dx, \quad \text{therefore} \quad \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| = x + c$$

so that

$$u(x) = \tanh(x + c).$$

Steady state solutions, example 4

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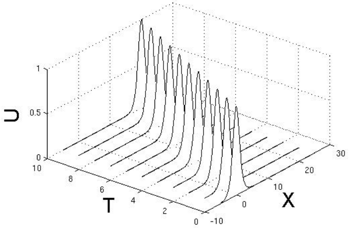
Solve by separation of variables:

$$u(x) = \frac{V}{2}\operatorname{sech}^2\left(\frac{\sqrt{V}}{2}(x+c)\right),$$



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Plugging into equation and keeping terms of order ϵ always gives a linear equation, called the *linearization* about $u_0(x)$.

- Nonlinear functions in equation must be (Taylor) expanded as series to identify order ϵ terms.
- One can study stability and dispersion of the linearization.
- This approximation becomes invalid when $w(x, t)$ becomes large enough.

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so that the linearization is now

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Flame-front propagation modeled by Kuramoto-Sivashinsky equation

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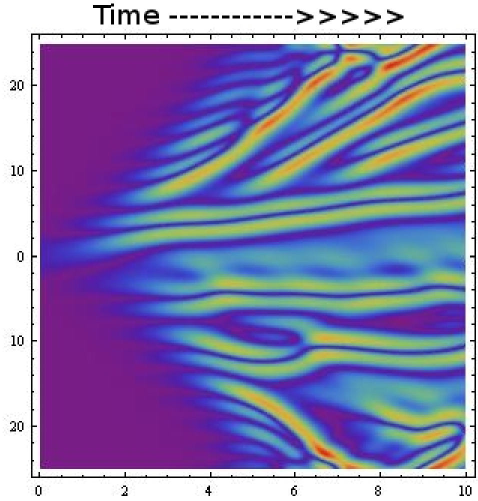
Dispersion relation of the form $w = \exp(\sigma t + ikx)$ gives

$$\sigma(k) = -k^4 + k^2.$$

Since $\sigma > 0$ for $|k| < 1$, $u = 0$ is unstable.

Example: Kuramoto-Sivashinsky simulation



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A thin liquid film of height $h(x, t)$ evolves according to the equation

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$$(h_0+\epsilon w)^3 = h_0^3+\epsilon 3h_0^2 w+\mathcal{O}(\epsilon^2), \quad (h_0+\epsilon w)^{-3} = h_0^{-3}-\epsilon 3h_0^{-4} w+\mathcal{O}(\epsilon^2).$$

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Inserting into equation,

$$\epsilon w_t = \left((h_0^3 + \epsilon 3h_0^2 w)[-\epsilon w_{xx} + h_0^{-3} - \epsilon 3Ah_0^{-4} w]_x\right)_x + \mathcal{O}(\epsilon^2),$$

so that retaining the ϵ size terms,

$$w_t = h_0^3(-w_{xxxx} - 3Ah_0^{-4} w_{xx}).$$

Example 3

A thin liquid film of height $h(x,t)$ evolves according to the equation

$$h_t = \left(h^3[-h_{xx} + Ah^{-3}]_x\right)_x,$$

where A describes intermolecular forces.
Linearize about a constant solution $h(x,t) = h_0$ by setting
$h(x,t) = h_0 + \epsilon w$ and Taylor expand

$$(h_0+\epsilon w)^3 = h_0^3+\epsilon 3h_0^2 w+\mathcal{O}(\epsilon^2), \quad (h_0+\epsilon w)^{-3} = h_0^{-3}-\epsilon 3h_0^{-4} w+\mathcal{O}(\epsilon^2).$$

Inserting into equation,

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The corresponding dispersion relation is found from
$w = \exp(\sigma t + ikx)$, giving

$$\sigma(k) = h_0^3(-k^4 + 3Ah_0^{-4} k^2),$$

Band of unstable wavenumbers $|k| < h_0^{-2}\sqrt{3A}$ if $A > 0$.

Example 4

Sine-Gordon equation is

$$u_{tt} = c^2 u_{xx} - \sin(u).$$

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For wave type equation, find dispersion relation $w(x,t) = \exp(ikx - i\omega t)$, giving

$$-\omega^2 = -c^2 k^2 - 1, \quad \omega(k) = \pm\sqrt{1 + c^2 k^2}.$$