The method of characteristics

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To find $u(x, t)$, go backward along these lines until $t = 0$, and use the initial condition:

$$u(x, t) = u(x - ct, 0) = f(x - ct)$$

This idea can be extended to many other transport-like equations.

Homogeneous transport equations

Consider

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If we choose $X'(T) = c(X(T),T)$, then

$$\frac{d}{dT}u(X(T),T) = c(X(T),T)u_x(X(T),T) + u_t(X(T),T) = 0$$

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The curves $X(T)$ that solve the ODE

$$X'(T) = c(X,T), \quad X(t) = x,$$

are *characteristics* which terminate at (x,t).

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whose solution is $X(T) = x \exp(T - t)$.

Since u is constant along the characteristic,

$$u(x, t) = u(X(0), 0) = \cos(xe^{-t}).$$

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Think of x as time variable and write equation as

$$u_x - \frac{x}{y} u_y = 0.$$

Characteristic curves $Y(X)$ solve separable equation

$$Y'(X) = -\frac{X}{Y}, Y(x) = y.$$

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Since the solution is constant along curves, setting $X = 0$ gives

$$u(x, y) = u(X, Y) = 2Y^2 = 2(x^2 + y^2).$$

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Let $U(T) = u(X(T),T)$ be solution u restricted to single characteristic,

$$U'(T) = g(U, X(T), T), \quad U(0) = u(X(0),0) = f(X(0)).$$

To find $u(x,t)$, go *backwards* along the characteristic until $T = 0$, and solve this ODE going *forwards* to $T = t$.

The general method of characteristics

Algorithm:

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1. Find the characteristic terminating at (x, t) by solving $X'(T) = c(X, T)$ subject to $X(t) = x$.
2. Find the solution along a characteristic by solving $U'(T) = g(U, X(T), T)$ subject to $U(0) = U(X(0), 0)$.
3. Find the solution at the endpoint of the characteristic by setting $u(x, t) = U(t)$.

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$$X(T) = e^{T-t}(x+t+1) - T - 1.$$

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Solution at (x,t) is

$$u(x,t) = U(t) = f(e^{-t}(x+t+1) - 1) + \frac{1}{2}t^2.$$

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Characteristics solve $X'(T) = 3$ with $X(t) = x$, so that $X = 3(T - t) + x$.

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Solution on characteristic evolves as $U'(T) = -U^2(T)$ with $U(0) = f(X(0)) = f(-3t + x)$.

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Separating $dU/U^2 = -dT$ leads to

$$U(T) = \frac{1}{T + 1/f(x - 3t)}.$$

Solution is obtained by setting

$$u(x,t) = U(t) = 1/[t + 1/f(x - 3t)].$$

Example 3.

Consider water flow on landscape of elevation $h(x)$.

Simple model: flow is downhill with magnitude $h'(x)$, which leads to flux $J = -h'(x)u$ and conservation equation

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Put in form of transport equation:

$$u_t - h'(x)u_x = h''(x)u.$$

Example 3,cont.

Valley described by $h(x) = x^2$, and initial depth is

$$u(x,0) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

so that equation is now

$$u_t - 2xu_x = 2u.$$

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Characteristics solve $X'(T) = -2X$ with terminal condition $X(t) = x$.

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Solution:

$$X(T) = xe^{2(t-T)}.$$

On characteristics, find u evolves according to $U' = 2U$ and

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Therefore $U(T) = e^{2T}$ if $|X(0)| = |xe^{2t}| < 1$, or zero otherwise. When $t = T$,

$$u(x,t) = U(t) = \begin{cases} e^{2t} & |x| \leq e^{-2t} \\ 0 & |x| > e^{-2t} \end{cases}$$