The method of characteristics

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To find u(x, t), go backward along these lines until t = 0, and use the initial condition:

$$u(x,t)=u(x-ct,0)=f(x-ct)$$

This idea can be extended to many other transport-like equations.

Consider

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If we choose X'(T) = c(X(T), T), then

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The curves X(T) that solve the ODE

$$X'(T) = c(X,T), \quad X(t) = x,$$

are *characteristics* which terminate at (x, t).

Solve $u_t + xu_x = 0$ with initial condition $u(x, 0) = \cos(x)$.

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whose solution is $X(T) = x \exp(T - t)$.

Since u is constant along the characteristic,

$$u(x, t) = u(X(0), 0) = \cos(xe^{-t}).$$

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, $u(0, y) = 2y^2$ for $y > 0$

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$$u_x - \frac{x}{y}u_y = 0.$$

Characteristic curves Y(X) solve separable equation

$$Y'(X) = -\frac{X}{Y}, Y(X) = y.$$

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Since the solution is constant along curves, setting X = 0 gives

$$u(x, y) = u(X, Y) = 2Y^2 = 2(x^2 + y^2).$$

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Let U(T) = u(X(T), T) be solution u restricted to single characteristic,

$$U'(T) = g(U, X(T), T), \quad U(0) = u(X(0), 0) = f(X(0)).$$

To find u(x, t), go backwards along the characteristic until T = 0, and solve this ODE going forwards to T = t.

The general method of characteristics

Algorithm:

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- Find the solution along a characteristic by solving U'(T) = g(U, X(T), T) subject to U(0) = U(X(0), 0).
- Find the solution at the endpoint of the characteristic by setting u(x,t) = U(t).

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Determine how *u* evolves on characteristic:

$$U'(T) = T$$
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Solution at (x, t) is

$$u(x,t) = U(t) = f(e^{-t}(x+t+1)-1) + \frac{1}{2}t^2.$$

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Solution on characteristic evolves as $U'(T) = -U^2(T)$ with U(0) = f(X(0)) = f(-3t + x).

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Separating $dU/U^2 = -dT$ leads to

$$U(T)=\frac{1}{T+1/f(x-3t)}.$$

Solution is obtained by setting

$$u(x, t) = U(t) = 1/[t + 1/f(x - 3t)].$$

Example 3.

Consider water flow on landscape of elevation h(x).

Simple model: flow is downhill with magnitude h'(x), which leads to flux J=-h'(x)u and conservation equation

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Put in form of transport equation:

$$u_t - h'(x)u_x = h''(x)u$$
.

Valley described by $h(x) = x^2$, and initial depth is

$$u(x,0) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

so that equation is now

$$u_t - 2xu_x = 2u$$
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Characteristics solve X'(T) = -2X with terminal condition X(t) = x. Solution:

$$X(T) = xe^{2(t-T)}$$
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On characteristics, find u evolves according to U' = 2U and

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Therefore $U(T) = e^{2T}$ if $|X(0)| = |xe^{2t}| < 1$, or zero otherwise. When t = T,

$$u(x,t) = U(t) = \begin{cases} e^{2t} & |x| \le e^{-2t} \\ 0 & |x| > e^{-2t} \end{cases}$$