Modeling using conservation laws

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Model ingredients:

- The flow or *flux* is a vector field $\mathbf{J}(x, t)$, so that $\mathbf{J} \cdot \hat{\mathbf{n}} dA$ is flow across infinitesimal area $\hat{\mathbf{n}} dA$.
- $Q(x, t)$ is the rate of inflow at point $\mathbf{x}$

Deriving a PDE for a conserved quantity

Conservation of $u(\mathbf{x}, t)$ on any region $R \subset \Omega$ implies

$$\frac{d}{dt}\int_R u\, d\mathbf{x} = \int_R \frac{\partial u}{\partial t}\, d\mathbf{x} = -\int_{\partial R} \mathbf{J}\cdot\hat{\mathbf{n}} d\mathbf{x} + \int_R Q(x,t) d\mathbf{x}.$$

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Since this is true for any subregion R, integrand is zero:

$$\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{J} = Q.$$

conservation form/continuity equation/transport equation

Boundary conditions for conserved quantities

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More specifically, if boundary in insulating, get "Neumann" boundary condition

$$\nabla u \cdot \hat{\mathbf{n}} = 0.$$

Remark: Dirichlet boundary condition $u = U(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega$ will not guarantee flux is zero at boundary.

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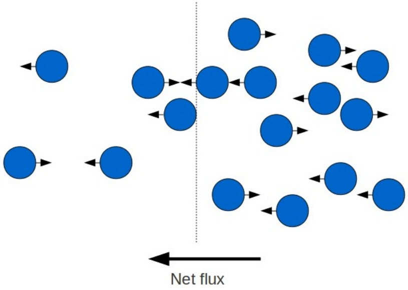
Traffic flow: speed can be modeled as a decreasing function of density $c = c_0 - mu$, so $J = u(c_0 - mu)$; conservation law becomes

$$u_t + c_0 u_x - m(u^2)_x = 0. \quad \text{(nonlinear transport equation)}$$

Example: Diffusion with a source

Random motions of particles (and other things) leads to *diffusion*.
Means that net flow has a direction toward regions of less density.



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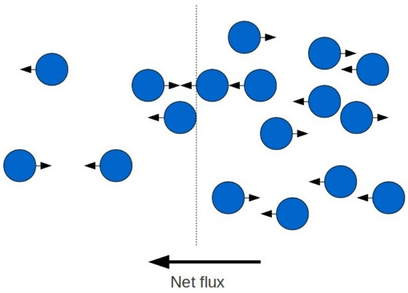

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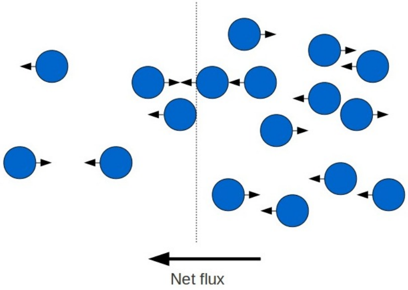

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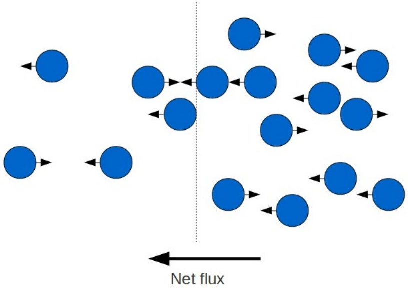
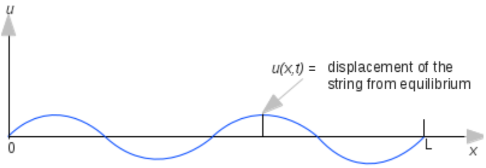


Simplest model: Fick/Fourier law $\mathbf{J} = -D\nabla u$. Sources $Q(\mathbf{x}, t)$ created by, for example heat production or chemical reactions. The conservation equation becomes

$$u_t = D\nabla\cdot\nabla u + Q = D\Delta u + Q, \quad \text{(diffusion equation)}$$

Example: wave equation



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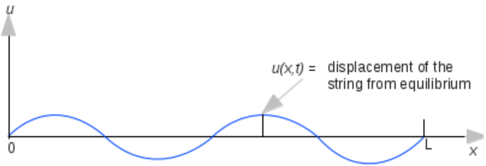

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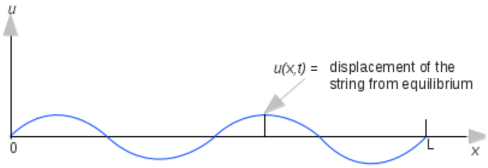

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Momentum conservation leads to

$$(u_t)_t = c^2 \nabla \cdot \nabla u = c^2 \Delta u, \quad \text{(wave equation)}$$

where $c^2 = \sigma/\rho$.

Steady state equations

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For a conservation law with flux $\mathbf{J}(u)$ and time independent source term Q, a *steady state solution* solves

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Steady state $u = u(x, y)$ solves

$$D\nabla\cdot\nabla u = \Delta u = Q(x, y).$$

If $Q \neq 0$, get *Poisson's equation*; if $Q \equiv 0$, get *Laplace's equation*.

Using conserved and dissipated quantities

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- Even if we could, often hard to see the essential aspects.

Idea: use coarse-grained quantities to study solutions qualitatively. Some of these are inspired by physics (energy, entropy), whereas others are completely abstract.

Conserved and dissipated quantities

A *functional* $F[u]$ maps u to the real numbers, e.g.

$$F[u] = \int_\Omega u(x)\,dx \quad \text{or} \quad F[u] = \int_\Omega |\nabla u|^2\,dx.$$

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Let $u(x,t) : \Omega \times [0,\infty) \to \mathbb{R}$ be a solution of some PDE, and suppose $F[u]$ has the form

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so that F can be regarded as depending on t.
Time evolution of $F[u]$ may be categorized as:

- If $dF/dt = 0$ for all u, then F is called *conserved*,
- If $dF/dt \leq 0$ for all u, then F is called *dissipated*.

Example: energy in the wave equation

Let u solve

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Yes, since $E(0) = 0$, $E(t) \equiv 0$, thus $u_x \equiv 0$. Using boundary conditions gives $u(x,t) \equiv 0$.
Converse also true: if $u(x,0) \neq 0$ initially, then solution never "dies out".

Example: heat equation

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One interpretation: arclength of x-cross sections of u can be approximated

$$\int_0^L \sqrt{1 + u_x^2} dx \approx \int_0^L 1 + \frac{1}{2} u_x^2 \, dx.$$

Since $dF/dt \le 0$, arclength diminishes and spatial oscillations die away.