The Fourier transform

For a function $f(x) : [-L, L] \to \mathbb{C}$, we have the orthogonal expansion

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \quad c_n = \frac{1}{2L} \int_{-L}^{L} f(y) e^{-in\pi y/L} dy.$$

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$F(k)$ is the *Fourier transform* of $f(x)$;
$f(x)$ is the *inverse transform* of $F(k)$.
Alternative notation: $F(k) = \hat{f}(k)$ and $f(x) = \check{F}(x)$.

A brief table of Fourier transforms

Description	Function	Transform
Delta function in x	$\delta(x)$	1
Delta function in k	1	$2\pi\delta(k)$
Exponential in x	$e^{-a\|x\|}$	$\frac{2a}{a^2+k^2}$
Exponential in k	$\frac{2a}{a^2+x^2}$	$2\pi e^{-a\|k\|}$
Gaussian	$e^{-x^2/2}$	$\sqrt{2\pi}e^{-k^2/2}$
Derivative in x	$f'(x)$	$ikF(k)$
Derivative in k	$xf(x)$	$iF'(k)$
Integral in x	$\int_{-\infty}^{x} f(x')dx'$	$F(k)/(ik)$
Translation in x	$f(x-a)$	$e^{-iak}F(k)$
Translation in k	$e^{iax}f(x)$	$F(k-a)$
Dilation in x	$af(ax)$	$F(k/a)$
Convolution	f(x)*g(x)	$F(k)G(k)$

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Example 1: The transform of $f''(x)$ is

$$\left[f''(x)\right]^{\wedge} = ik\left[f'(x)\right]^{\wedge} = (ik)^2\hat{f}(k) = -k^2\hat{f}(k).$$

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$$\left[\exp(-Ax^2)\right]^{\wedge} = \left[\exp(-[\sqrt{2A}x]^2/2)\right]^{\wedge} = \frac{1}{\sqrt{2A}}\left[\exp(-x^2/2)\right]^{\wedge}(k/\sqrt{2A}$$

$$= \sqrt{\frac{\pi}{A}}\exp(-[k/\sqrt{2A}]^2/2) = \sqrt{\frac{\pi}{A}}\exp(-k^2/(4A)).$$

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Example 3: Inverse transform of $e^{2ik}/(k^2+1)$

$$\left(\frac{e^{2ik}}{k^2+1}\right)^{\vee} = \left(\frac{1}{k^2+1}\right)^{\vee}(x+2) = \frac{1}{2}e^{-|x+2|}.$$

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Example 4: Inverse transform of $ke^{-k^2/2}$ is

$$\left(ke^{-k^2/2}\right)^\vee = -i\left(ike^{-k^2/2}\right)^\vee = -i\frac{d}{dx}\left(e^{-k^2/2}\right)^\vee$$

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$$= \frac{-i}{\sqrt{2\pi}}\frac{d}{dx}\left(\sqrt{2\pi}e^{-k^2/2}\right)^\vee = \frac{-i}{\sqrt{2\pi}}\frac{d}{dx}\left(e^{-x^2/2}\right) = \frac{ix}{\sqrt{2\pi}}e^{-x^2/2}.$$

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Change variables $z = x - y$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z)g(y)e^{-ik(y+z)}dydz$$

$$= \left(\int_{-\infty}^{\infty} f(z)e^{-ikz}dz\right)\left(\int_{-\infty}^{\infty} g(y)e^{-iky}dy\right) = \hat{f}(k)\hat{g}(k).$$

Divergent Fourier integrals as distributions

Since transform of $\delta(x)$ equals one

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk.$$

But integral does not converge! What does this mean?

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Idea: define the inverse transform more generally as a distribution which is the limit of nice integrals

$$f[\phi] = \lim_{L\to\infty} \int_{-\infty}^{\infty} f_L(x)\phi(x)dx, \quad f_L(x) = \frac{1}{2\pi} \int_{-L}^{L} \exp(ikx)F(k)dk$$

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$$f_L(x) = \frac{1}{2\pi}\int_{-L}^{L}\exp(ikx)\,dk = \frac{1}{\pi x}\sin(Lx).$$

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The limit $L\to\infty$ can be taken inside the integral, and

$$f[\phi] = \frac{\phi(0)}{\pi}\int_{-\infty}^{\infty}\frac{\sin(y)}{y}dy = \phi(0).$$