Solutions of differential equations using transforms

Process:

- Take transform of equation and boundary/initial conditions in one variable.
- Derivatives are turned into multiplication operators.
- Solve (hopefully easier) problem in k variable.
- Inverse transform to recover solution, often as a convolution integral.

Ordinary differential equations: example 1

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Just an algebraic equation, whose solution is

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Inverse transform of product of $\hat{f}(k)$ and $1/(1+k^2)$ is convolution:

$$u(x) = f(x) * \left(\frac{1}{1+k^2}\right)^{\vee} = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} f(y)dy.$$

But where was far field condition used?

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Inverse transform is

$$u(x) = \frac{C}{2\pi}\int_{-\infty}^{\infty} \exp(i[kx + k^3/3])dk.$$

With the choice $C = 1$ get the *Airy function*.

Partial differential equations, example 1

Laplace equation in upper half plane:

$$\begin{aligned} u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y > 0, \\ u(x,0) = g(x), \quad \lim_{y\to\infty} u(x,y) = 0. \end{aligned}$$

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Transform in the x variable only:

$$U(k,y) = \int_{-\infty}^{\infty} e^{-ikx} u(x,y) dx.$$

Note y-derivatives commute with the Fourier transform in x.

$$-k^2 U + U_{yy} = 0, \quad U(k,0) = \hat{g}(k), \quad \lim_{y\to\infty} U(k,y) = 0.$$

Partial differential equations, example 1, cont.

Now solve ODEs

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$$-k^2 U + U_{yy} = 0, \quad U(k,0) = \hat{g}(k), \quad \lim_{y\to\infty} U(k,y) = 0.$$

General solution is $U = c_1(k)e^{+|k|y} + c_2(k)e^{-|k|y}$. Using boundary conditions,

$$U(k,y) = \hat{g}(k)e^{-|k|y}.$$

Partial differential equations, example 1, cont.

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Inverse transform using convolution and exponential formulas

$$\begin{aligned} u(x,y) =& g(x) * \left(e^{-|k|y}\right)^{\vee} = g(x) * \left(\frac{y}{\pi(x^2+y^2)}\right) \\ =& \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yg(x_0)}{(x-x_0)^2+y^2} dx_0. \end{aligned}$$

Same formula as obtained by Green's function methods!

Partial differential equations, example 2

“Transport equation"

$$u_t + cu_x = 0, \quad -\infty < x < \infty, \quad t > 0, \qquad u(x,0) = f(x).$$

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$$U(k,t) = e^{-ickt}\hat{f}(k).$$

Use translation formula $f(x-a) = e^{-iat}\hat{f}(k)$ with $a = ct$,

$$u(x,t) = f(x-ct).$$

Partial differential equations, example 3

Consider the wave equation on the real line

$$u_{tt} = u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

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Solution of initial value problem

$$U(k,t) = \hat{f}(k)\cos(kt) + \frac{\hat{g}(k)}{k}\sin(kt).$$

Partial differential equations, example 3, cont.

Sines and cosines can be written in terms of complex exponentials

$$U(k,t) = \frac{1}{2}\hat{f}(k)(e^{ikt} + e^{-ikt}) + \frac{1}{2ik}\hat{g}(k)(e^{ikt} - e^{-ikt}).$$

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The inverse transform is now straightforward, using the exponential and integral formulas,

$$u(x,t) = \frac{1}{2}[f(x-t) + f(x+t)] + \frac{1}{2}\int_{-\infty}^{x} g(x'+t) - g(x'-t)dx'.$$

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Simplify integral using change of variables

$$\int_{-\infty}^{x} g(x'+t) - g(x'-t)dx' = \int_{-\infty}^{x+t} g(\xi)d\xi - \int_{-\infty}^{x-t} g(\xi)d\xi = \int_{x-t}^{x+t} g(\xi)d\xi.$$

All together get *d'Alembert's* formula

$$u(x,t) = \frac{1}{2}[f(x-t) + f(x+t)] + \frac{1}{2}\int_{x-t}^{x+t} g(\xi)d\xi.$$

Fundamental solutions

Consider generic, linear, time-dependent equation

$$u_t(x,t) = \mathcal{L}u(x,t),\ -\infty < x < \infty,\ u(x,0) = f(x), \lim_{|x|\to\infty} u(x,t) = 0,$$

where $\mathcal{L}$ is some operator(e.g. $\mathcal{L} = \partial^2/\partial x^2$).

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The *fundamental solution* $S(x, x_0, t)$ is a type of Green's function, solving

$$S_t = \mathcal{L}_x S,\ -\infty < x < \infty,\ S(x, x_0, 0) = \delta(x - x_0), \lim_{|x|\to\infty} S(x, x_0, t) = 0.$$

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Initial condition means S limits to a δ-function as $t \to 0$:

$$\lim_{t\to 0} \int_{-\infty}^{\infty} S(x, x_0, t)\phi(x)dx = \int_{-\infty}^{\infty} \delta(x - x_0)\phi(x)dx = \phi(x_0),$$

Fundamental solutions, cont.

Claim that the initial value problem has solution

$$u(x,t) = \int_{-\infty}^{\infty} S(x, x_0, t) f(x_0) dx_0,$$

Fundamental solutions, cont.

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$$u(x,t) = \int_{-\infty}^{\infty} S(x, x_0, t) f(x_0) dx_0,$$

Check:

$$u(x,0) = \lim_{t\to 0} \int_{-\infty}^{\infty} S(x, x_0, t) f(x_0) dx_0 = \int_{-\infty}^{\infty} \delta(x - x_0) f(x_0) dx_0 = f(x).$$

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Plugging u into the equation and moving time derivative inside the integral

$$u_t = \int_{-\infty}^{\infty} S_t(x,x_0,t) f(x_0) dx_0 = \int_{-\infty}^{\infty} \mathcal{L}_x S(x,x_0,t) f(x_0) dx_0.$$

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Now move operator outside integral

$$u_t = \mathcal{L}_x \int_{-\infty}^{\infty} S(x, x_0, t) f(x_0) dx_0 = \mathcal{L}_x u.$$

Fundamental solutions using the Fourier transform, example 1

For diffusion equation on the real line, S solves

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Solution to this ODE

$$\hat{S} = e^{-ix_0 k - Dk^2 t}.$$

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Inverse transform of

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uses translation, dilation, and Gaussian formulas:

$$S(x, x_0, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-(x-x_0)^2/(4Dt)}.$$

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It follows that the solution to $u_t = Du_{xx}$ and $u(x,0) = f(x)$ is

$$u(x,t) = \int_{-\infty}^{\infty} \frac{f(x_0)}{\sqrt{4\pi Dt}} e^{-(x-x_0)^2/(4Dt)} dx_0.$$

Fundamental solutions using the Fourier transform, example 2

Linearized Korteweg - de Vries (KdV) equation:

$$u_t = -u_{xxx}, \quad u(x,0) = f(x), \quad \lim_{|x|\to\infty} u(x,t) = 0.$$

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whose solution is $\hat{S}(k,x_0,t) = e^{-ix_0k}e^{ik^3t}$.

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whose solution is $\hat{S}(k,x_0,t) = e^{-ix_0k}e^{ik^3t}$.
Recall transform of Airy function $\mathrm{Ai}(x)$ is $e^{ik^3/3}$, therefore

$$S(x,x_0,t) = \left[e^{-ix_0k}e^{ik^3t}\right]^{\vee} = \left[e^{i(k/a)^3/3}\right]^{\vee}(x-x_0)$$
$$= a\mathrm{Ai}\Big(a(x-x_0)\Big), \quad a \equiv (3t)^{-1/3}.$$

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Solution to original equation:

$$u(x,t) = \frac{1}{(3t)^{1/3}}\int_{-\infty}^{\infty} \text{Ai}\left(\frac{x-x_0}{(3t)^{1/3}}\right) f(x_0)dx_0.$$

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- Inspiration: method of images. If $S_\infty(x; x_0, t)$ is the fundamental solution for the whole line, then:
  - Odd reflection $S = S_\infty(x; x_0, t) - S_\infty(x; -x_0, t)$ gives $S(0, \mathbf{x}_0, t) = 0$.
  - Even reflection $S = S_\infty(x; x_0, t) + S_\infty(x; -x_0, t)$ gives $S_x(0, \mathbf{x}_0, t) = 0$.

The method of images for fundamental solutions, example

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Use odd reflection of fundamental solution for whole line $S_\infty = e^{-(x-x_0)^2/(4Dt)}/\sqrt{4\pi Dt}$,

$$S(x, x_0, t) = \frac{1}{\sqrt{4\pi Dt}} \left[e^{-(x-x_0)^2/(4Dt)} - e^{-(x+x_0)^2/(4Dt)} \right]$$

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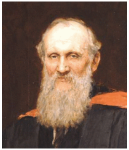
Therefore the solution u is just

$$u(x,t) = \int_0^\infty \frac{f(x_0)}{\sqrt{4\pi Dt}}\left[e^{-(x-x_0)^2/(4Dt)} - e^{-(x+x_0)^2/(4Dt)}\right] dx_0.$$

The age of the earth

Lord Kelvin: simple model of temperature of earth $u(x, t)$ at depth x and time t



$$u_t = Du_{xx},\ x > 0, \quad u(x, 0) = U_0,\ u(0, t) = 0.$$

Scale chosen so $u = 0$ on surface; assumes initially constant temperature (U_0) throughout the molten earth.

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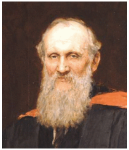

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$$u(x,t) = \int_0^\infty \frac{U_0}{\sqrt{4\pi Dt}} \left[e^{-(x-x_0)^2/(4Dt)} - e^{-(x+x_0)^2/(4Dt)} \right] dx_0.$$

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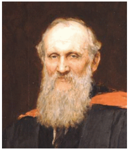

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Temperature gradient μ at surface is therefore

$$\mu = u_x(0,t) = \frac{U_0}{\sqrt{4\pi Dt}} \frac{1}{Dt} \int_0^\infty x_0 e^{-x_0^2/(4Dt)} dx_0 = \frac{U_0}{\sqrt{\pi Dt}}.$$

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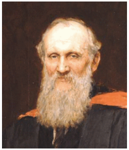

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Scale chosen so $u = 0$ on surface; assumes initially constant temperature (U_0) throughout the molten earth.

We found solution

$$u(x, t) = \int_0^\infty \frac{U_0}{\sqrt{4\pi Dt}} \left[e^{-(x-x_0)^2/(4Dt)} - e^{-(x+x_0)^2/(4Dt)} \right] dx_0.$$

Temperature gradient μ at surface is therefore

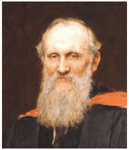
$$\mu = u_x(0, t) = \frac{U_0}{\sqrt{4\pi Dt}} \frac{1}{Dt} \int_0^\infty x_0 e^{-x_0^2/(4Dt)} dx_0 = \frac{U_0}{\sqrt{\pi Dt}}.$$

This relates the age of earth t to quantities we can estimate

$$U_0 \approx \text{melting temp. of iron} \approx 10^4 C, \quad D \approx 10^{-3} m^2/s, \quad \mu \approx 10^{-2} C/m,$$

The age of the earth

Lord Kelvin: simple model of temperature of earth $u(x,t)$ at depth x and time t



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which gives $t \approx 3 \times 10^7$ years !!??