Distributions and distributional derivatives

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Idea: create new class of function-like objects called *distributions* by defining how they “act" on smooth functions.

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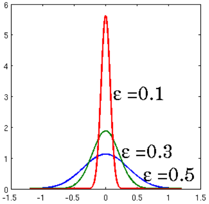
Approximate for $\epsilon > 0$

$$\delta_\epsilon(x) = \frac{1}{\epsilon\sqrt{\pi}} e^{-x^2/\epsilon^2}.$$



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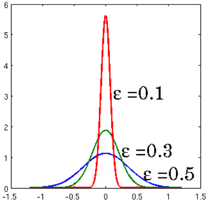

For any continuous and bounded $f(x)$,

$$\int_{-\infty}^{\infty} f(x)\delta_\epsilon(x-x_0)dx \approx f(x_0)\int_{-\infty}^{\infty}\delta_\epsilon(x-x_0)dx = f(x_0), \quad \epsilon \to 0.$$

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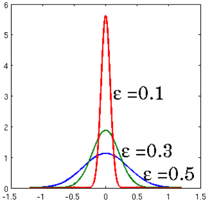

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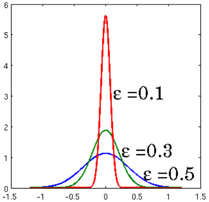

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Also define in higher dimensions:

$$\int_{\mathbb{R}^n} f(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}_0)d\mathbf{x} = f(\mathbf{x}_0), \quad \mathbf{x}, \mathbf{x}_0 \in \mathbb{R}^n.$$

Duality and linear functionals

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BOTH! We call this situation *duality*.

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- Distributions can be approximated by usual functions: there exists a sequence $d_n(x) \in \mathcal{D}$ so that

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- Distributions have integrals:

$$\int_{-\infty}^{\infty} d(x)\phi(x)dx \equiv d[\phi], \quad \text{for any } \phi \in \mathcal{D}.$$

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Ex. #1: Linear combinations of delta functions are distributions: If $d = 3\delta(x-1) + 2\delta(x)$ then corresponding linear functional is

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$$d[\phi] = x\phi(x)\Big|_0^\infty - \int_0^\infty \phi(x)dx = \int_{-\infty}^{\infty} \Big(-H(x)\Big)\phi(x)dx,$$

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Abuse of notation: $d = -H$.

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Ex. #2: What is the rule implied by the derivative of the δ-function? By definition, $\delta'[\phi] = -\delta[\phi'] = -\phi'(0)$.

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Now integrate by parts and integrate again:

$$\begin{aligned} f''[\phi] &= -\int_{-\infty}^{0} x\phi''(x)dx + \int_{0}^{\infty} x\phi''(x)dx \\ &= -x\phi'(x)|_{-\infty}^{0} + x\phi'(x)|_{0}^{\infty} + \phi(x)|_{-\infty}^{0} - \phi(x)|_{0}^{\infty} = 2\phi(0). \end{aligned}$$

So the second derivative of $|x|$ in the distributional sense is $2\delta(x)$.

Distributional derivatives in higher dimensions

In higher dimensions, distributional derivatives (gradients etc.) are defined using Green's identity: for a smooth function $u(x)$ and $\phi \in \mathcal{D}$, one has

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This motivates the definition of the *distributional Laplacian*:

$$(\Delta u)[\phi] = u[\Delta\phi] = \int_{\mathbb{R}^n} u \, \Delta\phi dx.$$

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We can now use Green's identity:

$$\Delta f[\phi] = \lim_{\epsilon\to 0}\int_{\partial B_\epsilon(0)} -\phi\frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{x}|}\right) + \frac{1}{|\mathbf{x}|}\frac{\partial\phi}{\partial n}dx$$

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The limit $\epsilon \to 0$ yields $\Delta f[\phi] = -4\pi\phi(0)$; therefore $\Delta f = -4\pi\delta(\mathbf{x})$.