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Physical interpretation: $G(\mathbf{x}, \mathbf{x}_0)$ is the influence at $\mathbf{x}$ of source at $\mathbf{x}_0$.

Relationship to the delta function

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whose solution is

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Find that the Green's function formally satisfies

$$\mathcal{L}_x G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0)$$

Reformulation of problem for $G(\mathbf{x}, \mathbf{x}_0)$

Split previous equation into two conditions:

$$\mathcal{L}_x G(\mathbf{x}, \mathbf{x}_0) = 0, \quad \text{when } \mathbf{x} \neq \mathbf{x}_0$$

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Integrating $\mathcal{L}_x G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0)$ over

$$\int_B \mathcal{L}_x G(\mathbf{x}, \mathbf{x}_0) dx = 1, \quad \text{for any ball } B \text{ centered at } \mathbf{x}_0.$$

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Lastly, G must satisfy the same type of homogeneous boundary conditions as u.

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$$u^{(n)}(x) + F(u^{(n-1)}(x), u^{(n-2)}(x), \ldots) = f(x),$$

where F is some expression involving lower order derivatives. The Green's function $G(x, x_0)$ formally satisfies

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Thus the $n - 1$-th derivative has a jump at x_0, and lower order derivatives are continuous:

$$\lim_{x \to x_0^+} \frac{\partial^{n-1} G}{\partial x^{n-1}} - \lim_{x \to x_0^-} \frac{\partial^{n-1} G}{\partial x^{n-1}} = 1, \quad \lim_{x \to x_0^+} \frac{\partial^m G}{\partial x^m} = \lim_{x \to x_0^-} \frac{\partial^m G}{\partial x^m}, m < n - 1.$$

Example: $\mathcal{L} = d^2/dx^2$

Suppose $u : \mathbb{R} \to \mathbb{R}$ solves

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$$G_{xx}(x, x_0) = 0 \text{ for } x \neq x_0, \quad G(0, x_0) = 0 = G(L, x_0),$$

with connection conditions

$$\lim_{x \to x_0^+} G_x(x, x_0) - \lim_{x \to x_0^-} G_x(x, x_0) = 1, \lim_{x \to x_0^+} G(x, x_0) = \lim_{x \to x_0^-} G(x, x_0).$$

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General solution to ODEs are:

$$G(x, x_0) = \begin{cases} c_1 x + c_3 & x < x_0 \\ c_2(x - L) + c_4 & x > x_0. \end{cases}$$

Imposing boundary conditions gives $c_3 = 0 = c_4$.

Example: $\mathcal{L} = d^2/dx^2$, cont.

For solution

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Solution to $u_{xx} = f, u(0) = 0 = u(L)$ in terms of G:

$$u(x) = \frac{1}{L}\left(\int_0^x x_0(x - L)f(x_0)dx_0 + \int_x^L x(x_0 - L)f(x_0)dx_0\right).$$

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Integrals evaluate to

$$-2\left(\frac{x^2}{2}(x-1) - x\frac{(x-1)^2}{2}\right) = x(1-x).$$

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General solution is $G = c_1 \exp(-kx) + c_2 \exp(kx)$; far-field conditions give

$$G(x, x_0) = \begin{cases} c_2 e^{kx} & x < x_0, \\ c_1 e^{-kx} & x > x_0. \end{cases}$$

Connection conditions imply

$$c_2 \exp(kx_0) = c_1 \exp(-kx_0), \quad -kc_1 \exp(-kx_0) - kc_2 \exp(kx_0) = 1,$$

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The entire Green's function may then be written compactly as

$$G(x, x_0) = -\exp(-k|x - x_0|)/2k,$$

and the solution is represented as

$$u(x) = -\frac{1}{2k} \int_{-\infty}^{\infty} f(x_0) e^{-k|x-x_0|} dx_0.$$

Using symmetry to obtain new Green's functions

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Insight: Subtract G_∞ and its reflection about $x = 0$:

$$G(x, x_0) = G_\infty(x, x_0) - G_\infty(-x, x_0)$$

This *does* satisfy $G(0, x_0) = 0$, but does it still solve correct equation?

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Solution is

$$u(x) = -\frac{1}{2k}\int_0^\infty f(x_0)\left[e^{-k|x-x_0|} - e^{-k|x+x_0|}\right] dx_0.$$

Dealing with inhomogeneous boundary conditions

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Idea to incorporate boundary conditions into representation formula:
For any u, v, integration by parts twice gives “Green's formula"

$$\int_0^L uv'' - vu'' \, dx = [uv' - vu']_0^L.$$

Dealing with inhomogeneous boundary conditions, cont.

To use the Green's formula

$$\int_0^L uv'' - vu'' \, dx = [uv' - vu']_0^L.$$

set $v(x) = G(x, x_0)$, giving

$$\int_0^L u(x)G_{xx}(x, x_0) - G(x, x_0)u''(x)dx = [u(x)G_x(x, x_0) - G(x, x_0)u'(x)]_{x=0}^{x=L}.$$

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Using $u_{xx}(x) = f(x)$, $G_{xx}(x, x_0) = \delta(x - x_0)$, $G(0, x_0) = 0 = G(L, x_0)$, this becomes

$$\begin{aligned} u(x_0) &= \int_0^L G(x, x_0)f(x)dx + [u(x)G_x(x, x_0)]_{x=0}^{x=L} \\ &= \int_0^L G(x, x_0)f(x)dx + BG_x(L, x_0) - AG_x(0, x_0). \end{aligned}$$