# Higher dimensional Green's functions

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Green's function formally solves $\Delta_x G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0)$, which is same as

$$\Delta_x G(\mathbf{x}, \mathbf{x}_0) = 0, \quad \mathbf{x} \neq \mathbf{x}_0, \quad \lim_{|\mathbf{x}|\to\infty} G(\mathbf{x}, \mathbf{x}_0) = 0.$$

with "normalization" condition

$$\int_{\partial B} \nabla_x G(\mathbf{x}, \mathbf{x}_0) \cdot \hat{n} dx = 1.$$

where $B$ is any ball with center $x_0$.

# Green's function in 3D, cont.

Symmetry allows $G = g(r)$, $r = |\mathbf{x} - \mathbf{x}_0|$, so that

$$\frac{1}{r^2}(r^2 g'(r))' = 0 \text{ if } r \neq 0, \quad \lim_{r\to\infty} g(r) = 0.$$

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Integrating twice,

$$g(r) = -\frac{c_1}{r} + c_2, \quad c_2 = 0 \text{ by far-field condition}$$

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Normalization: let $B$ be the unit sphere centered at $x_0$,

$$1 = \int_{\partial B} \nabla_x G(\mathbf{x}, \mathbf{x}_0) \cdot \hat{n}\, dx = \int_{\partial B} \frac{c_1}{r^2}\, dx = 4\pi c_1,$$

so that $c_1 = 1/4\pi$.

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so that $c_1 = 1/4\pi$.

The Green's function is therefore $G(\mathbf{x}, \mathbf{x}_0) = -1/(4\pi|\mathbf{x} - \mathbf{x}_0|)$ and

$$u(\mathbf{x}) = -\int_{\mathbb{R}^3} \frac{f(\mathbf{x}_0)}{4\pi|\mathbf{x} - \mathbf{x}_0|} dx_0^3.$$

## Example: $L = \Delta$ (two dimensions)

$$\Delta u = f, \quad \lim_{r\to\infty}\Big(u(r,\theta) - u_r(r,\theta) r \ln r\Big) = 0.$$

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Look for a Green's function of form $G = G(|\mathbf{x} - \mathbf{x}_0|) = g(r)$,

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Normalization condition (using $B =$ unit disk)

$$1 = \int_{\partial B} \nabla_x G(\mathbf{x}, 0) \cdot \hat{n} \, dx = \int_{\partial B} \frac{c_1}{r} \, dx = 2\pi c_1,$$

so that $c_1 = 1/2\pi$. Thus the Green's function is $G(\mathbf{x}, \mathbf{x}_0) = \ln|\mathbf{x} - \mathbf{x}_0|/2\pi$, and

$$u(\mathbf{x}) = \int_{\mathbb{R}^2} \frac{\ln|\mathbf{x} - \mathbf{x}_0| f(\mathbf{x}_0)}{2\pi} dx_0^2.$$

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As before, suppose $G(\mathbf{x}, \mathbf{x}_0) = g(r)$ where $r = |\mathbf{x} - \mathbf{x}_0|$, so that $\Delta G - G = \delta(\mathbf{x} - \mathbf{x}_0)$ is

$$g'' + \frac{1}{r} g' - g = 0, \quad r \neq 0, \qquad \lim_{r\to\infty} g(r) = 0,$$

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The Green’s function is therefore $G(\mathbf{x}, \mathbf{x}_0) = cK_0(|\mathbf{x} - \mathbf{x}_0|)$ where $c$ is found from a normalization condition

$$1 = \int_{\partial B_r(\mathbf{x}_0)} \frac{\partial_x G}{\partial n}(\mathbf{x}, \mathbf{x}_0)\, dx - \int_{B_r(\mathbf{x}_0)} G(\mathbf{x}, \mathbf{x}_0) dx \sim \int_{\partial B_r(\mathbf{x}_0)} \frac{\partial_x G}{\partial n}(\mathbf{x}, \mathbf{x}_0)\, dx,$$

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as $r \to 0$. It can be shown that $K_0 \sim -\ln(r)$ when $r$ is small, and therefore

$$\int_{\partial B_r(\mathbf{x}_0)} \frac{\partial_x G}{\partial n}(\mathbf{x}, \mathbf{x}_0)\, dx \sim -c \int_{\partial B_r(\mathbf{x}_0)} \frac{1}{r}\, dx = -2\pi c.$$

so that $c = -1/2\pi$.

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so that $c = -1/2\pi$. Therefore $u(\mathbf{x}) = -\int_{\mathbb{R}^2} \frac{K_0(|\mathbf{x}-\mathbf{x}_0|) f(\mathbf{x}_0)}{2\pi} dx_0$.