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But if u, v don't satisfy homogeneous boundary conditions, get

$$\int_\Omega (\mathcal{L}v)u \, d\mathbf{x} - \int_\Omega (\mathcal{L}u)v \, d\mathbf{x} = \text{boundary terms involving } u \text{ and } v.$$

This is called the *Green's formula*, which depends on $\mathcal{L}$ and Ω.

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Collapsing the integral involving the δ function,

$$u(\mathbf{x}_0) = \int_\Omega G(\mathbf{x}, \mathbf{x}_0)f(\mathbf{x})dx + \text{boundary terms}$$

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Simplifies to

$$u(\mathbf{x}_0) = \int_\Omega G(\mathbf{x}, \mathbf{x}_0) f(\mathbf{x}) dx + \int_{\partial\Omega} h(\mathbf{x})\nabla_x G(\mathbf{x}, \mathbf{x}_0) \cdot \hat{n}(x) \, dx,$$

Example: the Poisson integral formula revisited

In the case that Ω is a disk of radius a, Green's function is

$$G(r,\theta;r_0,\theta_0) = \frac{1}{4\pi}\ln\left(\frac{a^2}{r_0^2}\frac{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)}{r^2+a^4/r_0^2-2ra^2/r_0\cos(\theta-\theta_0)}\right).$$

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The boundary value problem

$$\Delta u = 0, \quad u(a,\theta) = h(\theta)$$

has a solution

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Need normal derivative of G

$$\begin{aligned}\nabla_x G(\mathbf{x},\mathbf{x}_0)\cdot\hat{n}(\mathbf{x}) &= G_r(r,\theta;r_0,\theta_0)\\ &= \frac{1}{4\pi}\left(\frac{2r-2r_0\cos(\theta-\theta_0)}{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)} - \frac{2rr_0^2-2r_0a^2\cos(\theta-\theta_0)}{r^2r_0^2+a^4-2rr_0a^2\cos(\theta-\theta_0)}\right),\end{aligned}$$

which at $r = a$ is

$$\frac{a}{2\pi}\left(\frac{1-(r/a)^2}{r^2+a^2-2ar\cos(\theta-\theta_0)}\right).$$

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Parameterize boundary integral using θ and $|dx| = a\,d\theta$,

$$u(r_0,\theta_0) = \frac{1}{2\pi}\int_0^{2\pi}\frac{(a^2-r_0^2)h(\theta)}{a^2+r_0^2-2ar_0\cos(\theta-\theta_0)}d\theta.$$

Neumann boundary conditions

Want to solve

$$\Delta u = 0, \quad \lim_{z \to \infty} u(x, y, z) = 0, \quad u_z(x, y, 0) = h(x, y),$$

in upper half space $\{(x, y, z) | z > 0\}$.

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$$\int_\Omega u \Delta v - v \Delta u \, dx = \int_{\partial\Omega} u \nabla v \cdot \hat{n} - v \nabla u \cdot \hat{n} \, dx.$$

has both Dirichlet and Neumann boundary terms in u, but only know $\nabla u(x, y, 0) \cdot \hat{n} = -u_z(x, y, 0)$.

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To make $\nabla v\cdot\hat{n} = \nabla G\cdot\hat{n}$ vanish on boundary, need Green's function to respect "boundary condition principle":

The Green's function must have the same type of boundary conditions as the problem to be solved, and they must be homogeneous.

Neumann boundary conditions,cont.

Method of images prescribes *even* reflection so $G_z = 0$ when $z = 0$:

$$G(x, y, z; x_0, y_0, z_0) = G_3(x, y, z; x_0, y_0, z_0) + G_3(x, y, z; x_0, y_0, -z_0)$$

$$= \frac{1}{4\pi}\left(\frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}} + \frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2}}\right)$$

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Substituting $v(\mathbf{x}) = G(\mathbf{x}, \mathbf{x}_0)$ into Green's formula and collapsing the δ-function integral,

$$u(\mathbf{x}_0) = -\int_{\partial\Omega} G(\mathbf{x}; \mathbf{x}_0) \nabla u(\mathbf{x}) \cdot \hat{n}(x)\, d\mathbf{x},$$

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Since $\hat{n}$ is directed *outward*, $\nabla u(\mathbf{x}) \cdot \hat{n}(x) = -u_z(x, y, 0)$, and

$$u(x_0, y_0, z_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{h(x, y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + z_0^2}} dxdy.$$

Symmetry (reciprocity) of the Green's function

If $\mathcal{L}$ is self-adjoint, might expect that its inverse to also be:

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Also: can interchange arguments of partial derivatives, e.g.

$$\begin{aligned}
\partial_x G(x, x_0) &= \lim_{h \to 0} (G(x + h, x_0) - G(x, x_0))/h \\
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For example, representation formula

$$u(\mathbf{x}_0) = \int_\Omega G(\mathbf{x}, \mathbf{x}_0) f(\mathbf{x}) dx + \int_{\partial\Omega} h(\mathbf{x}) \nabla_x G(\mathbf{x}, \mathbf{x}_0) \cdot \hat{n}(x)\, dx,$$

can be rewritten by exchanging the notation for $\mathbf{x}$ and $\mathbf{x}_0$ and using reciprocity,

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{x}_0) f(\mathbf{x}_0) dx_0 + \int_{\partial\Omega} h(\mathbf{x}_0) \nabla_{x_0} G(\mathbf{x}, \mathbf{x}_0) \cdot \hat{n}(\mathbf{x}_0)\, dx_0.$$