Dealing with boundaries and the method of images

Recall for domains $\Omega = \mathbb{R}^2, \mathbb{R}^3$, have “free space" Green’s functions for Poisson equation

$$G_2(\mathbf{x}; \mathbf{x}_0) = \frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}_0|$$
$$G_3(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|}.$$

Dealing with boundaries and the method of images

Recall for domains $\Omega = \mathbb{R}^2, \mathbb{R}^3$, have "free space" Green's functions for Poisson equation

$$G_2(\mathbf{x}; \mathbf{x}_0) = \frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}_0|$$
$$G_3(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|}.$$

In cases where there are boundaries, these don't satisfy boundary conditions!

Dealing with boundaries and the method of images

Recall for domains $\Omega = \mathbb{R}^2, \mathbb{R}^3$, have “free space" Green’s functions for Poisson equation

$$G_2(\mathbf{x}; \mathbf{x}_0) = \frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}_0|$$
$$G_3(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|}.$$

In cases where there are boundaries, these don’t satisfy boundary conditions!

Resolution: Use free space Green’s functions as particular solutions, or use them in conjunction with symmetric reflections

Arbitrary bounded domains

Consider Poisson equation for $u : \Omega \to \mathbb{R}$

$$\Delta u = f(x, y), \quad u = 0 \text{ on } \partial\Omega$$

where Ω is a bounded in $\mathbb{R}^2$.

Arbitrary bounded domains

Consider Poisson equation for $u : \Omega \to \mathbb{R}$

$$\Delta u = f(x, y), \quad u = 0 \text{ on } \partial\Omega$$

where Ω is a bounded in $\mathbb{R}^2$.
Need Green's function which satisfies

$$\Delta_x G = \delta(\mathbf{x} - \mathbf{x}_0), \quad G(\mathbf{x}, \mathbf{x}_0) = 0 \text{ when } x \in \partial\Omega.$$

Arbitrary bounded domains

Consider Poisson equation for $u : \Omega \to \mathbb{R}$

$$\Delta u = f(x, y), \quad u = 0 \text{ on } \partial\Omega$$

where Ω is a bounded in $\mathbb{R}^2$.
Need Green's function which satisfies

$$\Delta_x G = \delta(\mathbf{x} - \mathbf{x}_0), \quad G(\mathbf{x}, \mathbf{x}_0) = 0 \text{ when } x \in \partial\Omega.$$

Free space Green's function $G_2(\mathbf{x}; \mathbf{x}_0) = \ln|\mathbf{x} - \mathbf{x}_0|/2\pi$ satisfies right equation, but not boundary condition.

Arbitrary bounded domains

Consider Poisson equation for $u : \Omega \to \mathbb{R}$

$$\Delta u = f(x, y), \quad u = 0 \text{ on } \partial\Omega$$

where Ω is a bounded in $\mathbb{R}^2$.
Need Green's function which satisfies

$$\Delta_x G = \delta(\mathbf{x} - \mathbf{x}_0), \quad G(\mathbf{x}, \mathbf{x}_0) = 0 \text{ when } x \in \partial\Omega.$$

Free space Green's function $G_2(\mathbf{x}; \mathbf{x}_0) = \ln |\mathbf{x} - \mathbf{x}_0|/2\pi$ satisfies right equation, but not boundary condition.

Idea: use G_2 as a particular solution, and write $G(\mathbf{x}, \mathbf{x}_0) = G_2(\mathbf{x}; \mathbf{x}_0) + G_R(\mathbf{x}, \mathbf{x}_0)$. Then G_R solves

$$\Delta_x G_R = 0, \quad G_R(\mathbf{x}, \mathbf{x}_0) = -G_2(\mathbf{x}; \mathbf{x}_0) \text{ when } x \in \partial\Omega.$$

Arbitrary bounded domains

Consider Poisson equation for $u : \Omega \to \mathbb{R}$

$$\Delta u = f(x, y), \quad u = 0 \text{ on } \partial\Omega$$

where Ω is a bounded in $\mathbb{R}^2$.
Need Green's function which satisfies

$$\Delta_x G = \delta(\mathbf{x} - \mathbf{x}_0), \quad G(\mathbf{x}, \mathbf{x}_0) = 0 \text{ when } x \in \partial\Omega.$$

Free space Green's function $G_2(\mathbf{x}; \mathbf{x}_0) = \ln|\mathbf{x} - \mathbf{x}_0|/2\pi$ satisfies right equation, but not boundary condition.

Idea: use G_2 as a particular solution, and write $G(\mathbf{x}, \mathbf{x}_0) = G_2(\mathbf{x}; \mathbf{x}_0) + G_R(\mathbf{x}, \mathbf{x}_0)$. Then G_R solves

$$\Delta_x G_R = 0, \quad G_R(\mathbf{x}, \mathbf{x}_0) = -G_2(\mathbf{x}; \mathbf{x}_0) \text{ when } x \in \partial\Omega.$$

The function G_R is the *regular part* of the Green's function, and has no singular behavior.

Method of images

Main idea: If domain has certain symmetry, can use free space Green's functions as building blocks.

Method of images

Main idea: If domain has certain symmetry, can use free space Green's functions as building blocks.

Fundamental observation: for continuously differentiable $f(x) : \mathbb{R} \to \mathbb{R}$,

$$g(x) = f(x) - f(-x) \text{ is an odd function and } g(0) = 0,$$
$$h(x) = f(x) + f(-x) \text{ is an even function and } h'(0) = 0.$$

Method of images

Main idea: If domain has certain symmetry, can use free space Green's functions as building blocks.

Fundamental observation: for continuously differentiable $f(x) : \mathbb{R} \to \mathbb{R}$,

$$g(x) = f(x) - f(-x) \text{ is an odd function and } g(0) = 0,$$
$$h(x) = f(x) + f(-x) \text{ is an even function and } h'(0) = 0.$$

Therefore, sums/differences of free space Green's functions and their reflections can get boundary conditions right!

Example 1: upper half space

Consider problem for $u : \mathbb{R}^3 \cap \{z > 0\} \to \mathbb{R}$

$$\Delta u = f, \qquad \lim_{|\mathbf{x}| \to \infty} u(\mathbf{x}) = 0, \qquad u(x, y, 0) = 0.$$

Can't use free space Green's function

$$G_3(x, y, z; x_0, y_0, z_0) = -\frac{1}{4\pi\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}}$$

since it is not zero where $z = 0$.

Example 1: upper half space

Consider problem for $u : \mathbb{R}^3 \cap \{z > 0\} \to \mathbb{R}$

$$\Delta u = f, \quad \lim_{|\mathbf{x}|\to\infty} u(\mathbf{x}) = 0, \quad u(x, y, 0) = 0.$$

Can't use free space Green's function

$$G_3(x, y, z; x_0, y_0, z_0) = -\frac{1}{4\pi\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}}$$

since it is not zero where $z = 0$.

Using symmetry, however, suggests subtracting G from its mirror image about $z = 0$:

$$G(x, y, z; x_0, y_0, z_0) = G_3(x, y, z; x_0, y_0, z_0) - G_3(x, y, -z; x_0, y_0, z_0) =$$

$$\frac{1}{4\pi}\left(\frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}} - \frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2}}\right).$$

Example 1: upper half space, cont.

Check that

$$G(x, y, z; x_0, y_0, z_0) =$$

$$\frac{1}{4\pi}\left(\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}-\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}}\right).$$

solves the right problem.

Example 1: upper half space, cont.

Check that

$$G(x, y, z; x_0, y_0, z_0) =$$

$$\frac{1}{4\pi}\left(\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}-\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}}\right).$$

solves the right problem.

Easy to compute $\lim_{|\mathbf{x}|\to\infty} G = 0$, and also $G(x, y, 0; x_0, y_0, z_0) = 0$.

Example 1: upper half space, cont.

Check that

$$G(x, y, z; x_0, y_0, z_0) =$$

$$\frac{1}{4\pi}\left(\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} - \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}}\right).$$

solves the right problem.

Easy to compute $\lim_{|\mathbf{x}|\to\infty} G = 0$, and also $G(x, y, 0; x_0, y_0, z_0) = 0$.

Equation formally satisfied by G is

$$\Delta_x G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0) - \delta(\mathbf{x} - \mathbf{x}_0^*),$$

where $\mathbf{x}_0^* = (x_0, y_0, -z_0)$ is called the *image source*.

Example 1: upper half space, cont.

Check that

$$G(x, y, z; x_0, y_0, z_0) =$$
$$\frac{1}{4\pi}\left(\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} - \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}}\right).$$

solves the right problem.

Easy to compute $\lim_{|\mathbf{x}|\to\infty} G = 0$, and also $G(x, y, 0; x_0, y_0, z_0) = 0$.

Equation formally satisfied by G is

$$\Delta_x G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x}-\mathbf{x}_0) - \delta(\mathbf{x}-\mathbf{x}_0^*),$$

where $\mathbf{x}_0^* = (x_0, y_0, -z_0)$ is called the *image source*.

Extra δ-function is not a problem, since equation for G only needs to be satisfied in problem domain

$$\Delta_x G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x}-\mathbf{x}_0), \quad \text{for all } \mathbf{x} \in \mathbb{R}^3 \cap \{z > 0\} .$$

Example 2: disk

Want (in polar coordinates) $G(r, \theta; r_0, \theta_0)$ to solve $\Delta G = \delta(\mathbf{x} - \mathbf{x}_0)$ with boundary condition $G(a, \theta; r_0, \theta_0) = 0$.

Using free space G-function

$$G_2(r, \theta; r_o, \theta_0) = \frac{1}{4\pi} \ln(r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)).$$

Idea is to subtract G_2 from its “reflection" across the boundary, using image source at $\mathbf{x}_0^* = a^2\mathbf{x}_0/r_0^2$. The difference on the boundary is not zero, but is a constant $= -\ln(a/r_0)/(2\pi)$.

Example 2: disk

Want (in polar coordinates) $G(r, \theta; r_0, \theta_0)$ to solve $\Delta G = \delta(\mathbf{x} - \mathbf{x}_0)$ with boundary condition $G(a, \theta; r_0, \theta_0) = 0$.

Using free space G-function

$$G_2(r, \theta; r_o, \theta_0) = \frac{1}{4\pi} \ln(r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)).$$

Idea is to subtract G_2 from its “reflection" across the boundary, using image source at $\mathbf{x}_0^* = a^2\mathbf{x}_0/r_0^2$. The difference on the boundary is not zero, but is a constant $= -\ln(a/r_0)/(2\pi)$.
The desired Green's function is therefore

$$\begin{aligned} G(r, \theta; r_0, \theta_0) &= G_2(\mathbf{x}, \mathbf{x}_0) - G_2(\mathbf{x}, \mathbf{x}_0^*) + \frac{1}{2\pi} \ln(a/r_0) \\ &= \frac{1}{4\pi} \ln\left(\frac{a^2}{r_0^2} \frac{r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)}{r^2 + a^4/r_0^2 - 2ra^2/r_0 \cos(\theta - \theta_0)} \right). \end{aligned}$$

Example 2: disk,cont.

Check that

$$G(r,\theta;r_0,\theta_0) = \frac{1}{4\pi}\ln\left(\frac{a^2}{r_0^2}\frac{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)}{r^2+a^4/r_0^2-2ra^2/r_0\cos(\theta-\theta_0)}\right)$$
$$= G_2(\mathbf{x},\mathbf{x}_0) - G_2(\mathbf{x},\mathbf{x}_0^*) + \frac{1}{2\pi}\ln(a/r_0).$$

satisfies the correct conditions:

Example 2: disk,cont.

Check that

$$G(r,\theta;r_0,\theta_0) = \frac{1}{4\pi}\ln\left(\frac{a^2}{r_0^2}\frac{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)}{r^2+a^4/r_0^2-2ra^2/r_0\cos(\theta-\theta_0)}\right)$$
$$= G_2(\mathbf{x},\mathbf{x}_0) - G_2(\mathbf{x},\mathbf{x}_0^*) + \frac{1}{2\pi}\ln(a/r_0).$$

satisfies the correct conditions:

Note $\Delta G = \delta(\mathbf{x}-\mathbf{x}_0) - \delta(\mathbf{x}-\mathbf{x}_0^*)$, which is which $\Delta G = \delta(\mathbf{x}-\mathbf{x}_0)$ when restricted to the disk.

Example 2: disk,cont.

Check that

$$G(r,\theta;r_0,\theta_0) = \frac{1}{4\pi}\ln\left(\frac{a^2}{r_0^2}\frac{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)}{r^2+a^4/r_0^2-2ra^2/r_0\cos(\theta-\theta_0)}\right)$$
$$= G_2(\mathbf{x},\mathbf{x}_0) - G_2(\mathbf{x},\mathbf{x}_0^*) + \frac{1}{2\pi}\ln(a/r_0).$$

satisfies the correct conditions:

Note $\Delta G = \delta(\mathbf{x}-\mathbf{x}_0) - \delta(\mathbf{x}-\mathbf{x}_0^*)$, which is which $\Delta G = \delta(\mathbf{x}-\mathbf{x}_0)$ when restricted to the disk.

Also, evaluating G on the boundary,

$$G(a,\theta;r_o,\theta_0) = \frac{1}{4\pi}\ln\left(\frac{a^2}{r_0^2}\frac{a^2+r_0^2-2ar_0\cos(\theta-\theta_0)}{a^2+a^4/r_0^2-2a^3/r_0\cos(\theta-\theta_0)}\right)$$
$$= \frac{1}{4\pi}\ln\left(\frac{a^2+r_0^2-2ar_0\cos(\theta-\theta_0)}{r_0^2+a^2-2ar_0\cos(\theta-\theta_0)}\right) = 0.$$