Elements of linear algebra

A vector space S is a set (numbers, vectors, functions) which has addition and scalar multiplication defined, so that the *linear combination*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$$

is also a member of S. The set of all such linear combinations is the *span* of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

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A set of elements $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is *linearly independent* if any one element is not the linear combination of the others. Informally, this says that a *linearly dependent* set is redundant, in the sense that some subset would have exactly the same span.

Linear operators in finite dimensions.

Suppose $\mathbf{A}$ is a $n \times n$ matrix, and $\mathbf{v}$ is a n-dimensional vector. The matrix-vector product $\mathbf{y} = A\mathbf{v}$ can be regarded as a mapping

$$\mathbf{v} \to A\mathbf{v}.$$

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Linearity property: a transformation of a linear combination is the linear combination of the linear transformations. For the matrix A this means that

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More generally: if f_1, f_2 are elements of a vector space S, then a linear operator $\mathcal{L}$ is a mapping $S \to S'$ so that

$$\mathcal{L}(c_1 f_1 + c_2 f_2) = c_1 \mathcal{L} f_1 + c_2 \mathcal{L} f_2.$$

Eigenvalues and eigenvectors in finite dimensions

Recall that $\mathbf{v} \neq 0, \lambda$ is an eigenvector-eigenvalue pair of $n \times n$ matrix $\mathbf{A}$ if

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Usually there are exactly n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ This is useful since eigenvectors form a basis

$$\mathbf{x} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \ldots + c_n \mathbf{v}_n, \quad \text{for any } \mathbf{x} \in \mathbb{R}$$

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Applying $\mathbf{A}$ to the vector $\mathbf{x}$

$$\begin{aligned}\mathbf{A}\mathbf{x} =& \mathbf{A}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n) \\ =& c_1\mathbf{A}\mathbf{v}_1 + c_2\mathbf{A}\mathbf{v}_2 + \ldots + c_n\mathbf{A}\mathbf{v}_n \\ =& c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \ldots + c_n\lambda\mathbf{v}_n.\end{aligned}$$

Thus, *eigenvectors decompose a linear operator into a linear combination.*

Inner products

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1. Symmetry: $\langle v, u \rangle = \langle u, v \rangle$ for every $u, v \in S$
   (Complex version: $\langle v, u \rangle = \overline{\langle u, v \rangle}$)
2. Linearity in first variable: for any vector $\mathbf{v}$ and vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ we have

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$\langle \mathbf{v}, \mathbf{v} \rangle = |\mathbf{v}|^2 =$ "length" squared.
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$ means $\mathbf{v}_1, \mathbf{v}_2$ are *orthogonal*

Adjoint operators

For a linear operator $\mathcal{L}$, the adjoint $\mathcal{L}$ with respect to a given inner product is some other operator $\mathcal{L}^\dagger$ so that

$$\langle \mathcal{L}\mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \mathbf{v}_1, \mathcal{L}^\dagger \mathbf{v}_2 \rangle, \quad \text{for all } \mathbf{v}_1, \mathbf{v}_2$$

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By the usual rules of matrix multiplication

$$\begin{aligned} \langle A\mathbf{v}_1, \mathbf{v}_2 \rangle &= (\mathbf{A}\mathbf{v}_1) \cdot \mathbf{v}_2 = \mathbf{v}_2^T \mathbf{A}\mathbf{v}_1 = (\mathbf{v}_2^T \mathbf{A}\mathbf{v}_1)^T \\ &= \mathbf{v}_1^T \mathbf{A}^T \mathbf{v}_2 = \mathbf{v}_1 \cdot (\mathbf{A}^T \mathbf{v}_2) = \langle \mathbf{v}_1, A^T \mathbf{v}_2 \rangle, \end{aligned}$$

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Caution: there are many different inner products, and the adjoint depends on them.

Self-adjointness

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Consider a linear operator represented by a matrix A. If it is self-adjoint, then

1. The eigenvalues of A are real.
2. The eigenvectors of A are orthogonal to one another.

Similar statements can be made for eigenvalue problems of differential operators.

Orthogonal expansions

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be eigenvectors of a self-adjoint matrix A.
The linear combination

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Thus

$$c_k = \frac{\langle \mathbf{x}, \mathbf{v}_k \rangle}{\langle \mathbf{v}_k, \mathbf{v}_k \rangle}.$$

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Remark: If eigenvectors are *normalized*, $\langle \mathbf{v}_k, \mathbf{v}_k \rangle = 1$ and $c_k = \langle \mathbf{x}, \mathbf{v}_k \rangle$

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A *differential operator* is a linear operator composed of derivatives, which acts on a vector space of functions.

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Example 1: d^2/dx^2 maps $f(x)$ to $f''(x)$, and satisfies linearity

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Example 2: $\mathcal{L}f = \big(g(x)f'(x)\big)'$ is linear, since

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Technical point: Set of functions which constitute a vector space matters. For example, adjoints, eigenvalues, etc. depend on boundary conditions which define which space is being used.

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Some function space notation: $C^k(\Omega)$ = space of k-times continuously differentiable functions with domain Ω.
$C_0^\infty(\Omega)$ = space of infinitely differentiable functions whose value on the boundary of Ω are zero.

The superposition principle

All linear differential equations for unknown f have form

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\begin{aligned}
\mathcal{L}f &= 0, \quad \text{(homogeneous equations)} \\
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If f_p is a *particular solution* solving $\mathcal{L}f_p = g(x)$ then $h = f - f_p$ solves a homogeneous equation $\mathcal{L}h = 0$.

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Strategy for solving inhomogeneous equations: (1) Find particular solution, (2) find general solution $\mathcal{L}h = 0$, (3) total solution is $f = f_p + h$.

The superposition principle and boundary conditions

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Extended superposition principle:
If $f_1, f_2, \ldots$ satisfy linear, homogeneous boundary conditions, then so does a linear combination $c_1 f_1 + c_2 f_2 + \ldots$