Sturm-Liouville operators

Sturm-Liouville operators have form (given $p(x) > 0, q(x)$)

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For example,

$$\mathcal{L} = \frac{d^2}{dx^2} - 4, \quad \mathcal{L}f(x) = f''(x) - 4f(x).$$

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Typical vector space of functions $\mathcal{L}$ acts on: $C_0^\infty[a, b]$ so that $f(a) = f(b) = 0$.

Inner products and self adjointness

One possible inner product for $C_0^\infty[a, b]$ is

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Adjoint of Sturm-Liouville operator: compute by moving operator around using integration by parts. For any two functions f, g in $C_0^\infty[a,b]$,

$$\begin{aligned}
\langle \mathcal{L}f, g \rangle &= \int_a^b \frac{d}{dx}\Big(p(x)\frac{df}{dx}\Big)g(x) + q(x)f(x)g(x)dx \\
&= \int_a^b -p(x)\frac{df}{dx}\frac{dg}{dx} + q(x)f(x)g(x)dx \\
&= \int_a^b \frac{d}{dx}\Big(p(x)\frac{dg}{dx}\Big)f(x) + q(x)f(x)g(x)dx \\
&= \langle f, \mathcal{L}g \rangle.
\end{aligned}$$

Thus S-L operators are self-adjoint on $C_0^\infty[a,b]$.

More examples of operator adjoints

Example 2: Use inner product on $C_0^\infty[0, R]$

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For linear operator $\mathcal{L}f = r^{-1}(rf')'$ integration by parts gives

$$\begin{aligned}
\langle \mathcal{L}f, g\rangle &= \int_0^R r\left[r^{-1}\frac{d}{dr}\left(r\frac{df}{dr}\right)\right] g(r) dr \\
&= -\int_0^R r\frac{df}{dr}\frac{dg}{dr} dr, \quad \text{(IBP)} \\
&= \int_0^R \frac{d}{dr}\left(r\frac{dg}{dr}\right) f(r)\, dr, \quad \text{(IBP again)} \\
&= \int_0^R r\left[r^{-1}\frac{d}{dr}\left(r\frac{dg}{dr}\right)\right] f(r)\, dr \\
&= \langle f, \mathcal{L}g\rangle.
\end{aligned}$$

Thus $\mathcal{L}$ is self adjoint with respect to the weighted inner product (but not with respect to the L^2 one!)

More examples of operator adjoints

Example 3: $\mathcal{L} = d/dx$ acting on $C_0^\infty[a,b]$.

$$\langle \mathcal{L}f, g\rangle = \int_a^b \frac{df}{dx} g(x) = -\int_a^b \frac{dg}{dx} f(x) dx = \langle f, -\mathcal{L}g\rangle.$$

Therefore $\mathcal{L}^\dagger = -\mathcal{L}$.

Eigenvalue problems for differential operators

Problem: find *eigenfunction* $v(x)$ and eigenvalue λ solving

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Remarks:

- Sometimes written $\mathcal{L}v(x) + \lambda v(x) = 0$
- Constant times $v(x)$ is also eigenfunction.
- Infinite dimensions implies infinite number of eigenvalues and eigenfunctions (usually).
- If eigenfunctions span required vector space, called *complete*.

Sturm-Liouville eigenvalue problems

Self-adjoint operators (like S-L) have nice properties:

1. The eigenvalues are real.
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Facts about this problem:

1. The real eigenvalues can be ordered $\lambda_1 < \lambda_2 < \lambda_3 \ldots$ so that there is a smallest (but not largest) eigenvalue.
2. The eigenfunctions $v_n(x)$ corresponding to each eigenvalue $\lambda_n(x)$ form a complete set, i.e. any $f \in C_0^\infty[a, b]$, we can write f as a (infinite) linear combination

$$f = \sum_{n=1}^{\infty} c_n v_n(x).$$

Example: A Sturm-Liouville eigenvalue problem

Consider operator $\mathcal{L} = d^2/dx^2$ on the vector space $C_0^\infty[0, \pi]$. The eigenvalue problem reads

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If $\lambda > 0$, ODE has general solution

$$v(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

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Then other b.c. implies $B\sin(\pi\sqrt{\lambda}) = 0$ so $\pi\sqrt{\lambda}$ is a multiple of π or

$$\lambda_n = n^2, \quad n = 1, 2, 3, \ldots.$$

The corresponding eigenfunctions are

$$v_n(x) = \sin(\sqrt{\lambda}x) = \sin(nx), n = 1, 2, 3, \ldots$$

Example: A Sturm-Liouville eigenvalue problem, cont.

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Example: A Sturm-Liouville eigenvalue problem, cont.

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Case $\lambda < 0$ gives $v(x) = A\exp(\sqrt{|\lambda|}x) + B\exp(-\sqrt{|\lambda|}x)$.
Boundary conditions imply $A + B = 0$ and
$A\exp(\sqrt{|\lambda|}\pi) + B\exp(-\sqrt{|\lambda|}\pi) = 0$ or

$$\begin{pmatrix} 1 & 1 \\ \exp(\sqrt{|\lambda|}\pi) & \exp(-\sqrt{|\lambda|}\pi) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

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Determinant is $\exp(\sqrt{|\lambda|}\pi) - \exp(-\sqrt{|\lambda|}\pi) \neq 0$ so that the only solution is $A = B = 0$.

Big payoff: completeness of eigenfunctions means that any smooth function with $f(0) = 0 = f(\pi)$ can be written

$$f(x) = \sum_{n=1}^{\infty} B_n \sin(nx), \quad \text{“Fourier sine series”}$$

Fourier series

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Computing coefficients in an orthogonal expansion is just a matter of taking inner products:

$$B_n = \frac{\langle f(x), \sin(nx) \rangle}{\langle \sin(nx), \sin(nx) \rangle} = \frac{\int_0^\pi f(x) \sin(nx) dx}{\int_0^\pi \sin^2(nx) dx} = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx) dx.$$

Other Fourier series

Series type	Space of functions	Orthogonal expansion for $f(x)$	Coefficients
Fourier	$f(x) : [-L, L] \to \mathbb{R}$ $f(-L) = f(L)$ $f'(-L) = f'(L)$	$\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{L})$ $+ \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{L})$	$A_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(\frac{n\pi x}{L}) dx$ $B_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(\frac{n\pi x}{L}) dx$
Sine	$f(x) : [0, L] \to \mathbb{R}$ $f(0) = 0 = f(L)$	$\sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{L})$	$B_n = \frac{2}{L} \int_{-L}^{L} f(x) \sin(\frac{n\pi x}{L}) dx$
Cosine	$f(x) : [0, L] \to \mathbb{R}$ $f'(0) = 0 = f'(L)$	$\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{L})$	$A_n = \frac{2}{L} \int_{-L}^{L} f(x) \cos(\frac{n\pi x}{L}) dx$
Complex	$f(x) : [-L, L] \to \mathbb{C}$ $f(-L) = f(L)$ $f'(-L) = f'(L)$	$\sum_{n=-\infty}^{\infty} c_n \exp(\frac{in\pi x}{L})$	$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) \exp(\frac{-in\pi x}{L}) dx$

Linear integral operators

"Hilbert-Schmidt" integral operators

$$\mathcal{L}u(x) = \int_\Omega k(x,y)u(x)dx.$$

obey linearity

$$\begin{aligned}\mathcal{L}(c_1u_1 + c_2u_2) &= c_1\int_\Omega k(x,y)u_1(x)dx + c_2\int_\Omega k(x,y)u_2(x)dx \\ &= c_1\mathcal{L}u_1 + c_2\mathcal{L}u_2.\end{aligned}$$

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- Often inverses of differential operators: $\mathcal{L}u = g$ has solution $u = \mathcal{L}^{-1}g$ where $\mathcal{L}^{-1}$ is HS type
- Adjoints are similar. Example: using the L^2 inner product

$$\begin{aligned}\langle \mathcal{L}u, v\rangle &= \int_{\Omega} v(y) \int_{\Omega} k(x, y)u(x)\, dxdy \\ &= \int_{\Omega} u(x) \int_{\Omega} k(x, y)v(y)\, dydx = \langle u, \mathcal{L}^{\dagger} v\rangle\end{aligned}$$

where $\mathcal{L}^{\dagger}$ is an integral operator with kernel $k(y, x)$.