Problems with three independent variables

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$$\begin{aligned} u_t &= \Delta u \quad (\textit{Diffusion}) \\ u_{tt} &= \Delta u \quad (\textit{Wave}) \\ -u_{zz} &= \Delta u \quad (\textit{Laplace}) \end{aligned}$$

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Will need homogeneous boundary conditions such as

$$\begin{aligned} u(x, y, \cdot) &= 0, \quad (x, y) \in \partial D \quad (\textit{Dirichlet}) \\ \nabla u(x, y, \cdot) \cdot \hat{\mathrm{n}} &= 0, \quad (x, y) \in \partial D \quad (\textit{Neumann}) \end{aligned}$$

On the other hand, conditions at $t = 0$ or $z = a, b$ are arbitrary.

Separating variables

Look for solutions of form $u = T(t)v(x,y)$ or $u = Z(z)v(x,y)$

$$\begin{aligned}
\frac{T'}{T} &= \tfrac{\Delta v}{v} &= -\lambda \quad (\mathit{Diffusion}) \\
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$$\Delta v + \lambda v = 0, \quad \text{plus boundary conditions.}$$

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With suitable boundary conditions

- eigenvalues are real, non-negative
- Eigenfunctions are orthogonal w.r.t. inner product $\langle u, v \rangle = \int_D uv\, dx$.

Solution in terms of eigenfunctions and eigenvalues

Solving the ODEs for the T and Z variables and taking a superposition, we arrive at the general solutions

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\begin{aligned}
u(x,y,t) &= \sum_{n=1}^{\infty} A_n \exp(-\lambda_n t) v_n(x,y) \quad (\textit{Diffusion}) \\
u(x,y,t) &= \sum_{n=1}^{\infty} [A_n \cos(\sqrt{\lambda}_n t) + B_n \sin(\sqrt{\lambda}_n t)] v_n(x,y) \quad (\textit{Wave}) \\
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- Main issue: solve the eigenvalue problem.
- Difficult to write complete solution for arbitrary domain D.
- Three tractable cases are where D is a rectangle, a disk, and the surface of a sphere.

Green's identity

Let $u, v : D \to \mathbb{R}$ be smooth functions. Apply the divergence theorem to $u \nabla v$,

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Use $\nabla\cdot(u\nabla v) = \nabla u \cdot \nabla v + u\Delta v$,

$$\int_D u\Delta v d\mathrm{x} = -\int_D \nabla u \cdot \nabla v d\mathrm{x} + \int_{\partial D} u\nabla v \cdot \hat{\mathrm{n}}d\mathrm{x}.$$

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Remark: just like integration by parts in higher dimensions.

Self-adjointness of the Laplacian

Consider space of smooth functions with domain D, satisfying either Dirichlet or Neumann homogeneous boundary conditions. Use inner product

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- The integrals on the boundary ∂D vanish because of the boundary conditions.
- It follows Laplacian is self-adjoint.

Non-negativity of the eigenvalues

Take inner product of an eigenfunction v with both sides of the the eigenvalue equation $\mathcal{L}v + \lambda v = 0$, leading to

$$\lambda = -\frac{\langle \mathcal{L}v, v\rangle}{\langle v, v\rangle}, \quad \text{“Rayleigh quotient”}$$

Does not determine λ, but can be used to estimate it!

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- Expression on the right is non-negative.
- $\lambda = 0$ can be zero only with Neumann boundary conditions.