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- Three tractable domains: rectangle, disk, surface of a sphere.

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Eigenfunctions on the rectangle

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By separation principle terms on left are constants $-\lambda_x, -\lambda_y$, so that

$$X'' + \lambda_x X = 0, \quad Y'' + \lambda_y Y = 0, \quad \lambda = \lambda_x + \lambda_y.$$

Eigenfunctions on the rectangle, cont.

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Therefore all possible combinations are

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Many other possible boundary conditions; for example, Neumann leads to $\nabla v \cdot \hat{\mathbf{n}} = 0$ lead to

$$v_{nm}(x, y) = \cos(nx)\cos(my), \quad \lambda_{nm} = n^2 + m^2, \quad n, m = 0, 1, 2, 3, \ldots.$$

Nodes of the eigenfunctions

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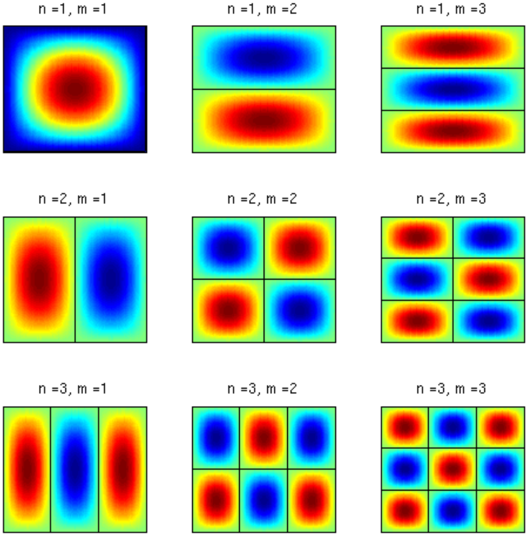

Eigenfunctions on the disk

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Look for solutions $v = R(r)\Theta(\theta)$ and separate variables

$$\frac{r^2 R'' + rR' + \lambda r^2 R}{R} + \frac{\Theta''}{\Theta} = 0.$$

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Boundary conditions for Θ are 2π-periodic, so $\lambda_\theta = n^2$ and

$$\Theta = \begin{cases} \cos(n\theta), & n = 0, 1, 2, \dots \\ \sin(n\theta), & n = 1, 2, 3, \dots. \end{cases}$$

Eigenfunctions on the disk,cont.

Using $\lambda_\theta = n^2$, equation for radial part $R(r)$ is

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If $\lambda > 0$, simplify using change of variables $\rho = \sqrt{\lambda} r$ so $R = R(\rho)$ solves

$$\rho^2 R'' + \rho R' + (\rho^2 - n^2)R = 0, \quad \text{(Bessel's equation)}.$$

Eigenfunctions on the disk,cont.

Approximate for ρ small gives Euler's equation

$$\rho^2 R'' + \rho R' - n^2 R = 0$$

Therefore for each n expect two linearly independent solutions $J_n(\rho), Y_n(\rho)$

$$J_n(\rho) \sim \begin{cases} \rho^n & n > 0 \\ 1 & n = 0, \end{cases} \qquad Y_n(\rho) \sim \begin{cases} \rho^{-n} & n > 0 \\ \ln \rho & n = 0, \end{cases}$$

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- $J_n(\rho)$ and $Y_n(\rho)$ are the *Bessel functions*.
- Bessel functions have an infinite number of zeros; label them

$$\beta_{nm} = m\text{th positive zero of } J_n(\rho).$$

Eigenfunctions on the disk,cont.

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What *are* the Bessel functions?

To find a solution to Bessel's equation, use "method of Frobenius":

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For $k = 0, 1$,

$$[\alpha^2 - n^2] a_0 = 0, \quad [(\alpha + 1)^2 - n^2] a_1 = 0.$$

Thus $\alpha = \pm n$ and $a_1 = 0$ (this leads to $a_3, a_5, a_7, \ldots = 0$).

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Other coefficients solve recursion relation $a_k = -\frac{a_{k-2}}{(\alpha+k)^2-n^2}$; with $\alpha = n$ and $a_0 = 2^{-n}/n!$, giving

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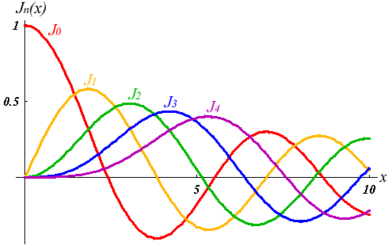
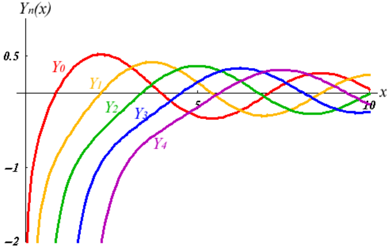
Finally, get convergent (for all ρ) power series

$$J_n(\rho) = \sum_{k=0}^{\infty} (-1)^k \frac{(\rho/2)^{n+2k}}{k!(n+k)!}$$

Bessel functions in pictures





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Thus the eigenfunctions (for each n) are orthogonal:

$$\langle J_n(\beta_{nm} r/a), J_n(\beta_{nk} r/a)\rangle = 0, \quad m \neq k.$$

Nodes of the eigenfunctions on a disk

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