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- Look for separated solutions $u = f(v_1)g(v_2)$
- Superpositions of separated solutions will give entire solution.
- Other boundary/initial conditions will determine coefficients of superposition.

The separation principle

Suppose we have independent variables $x_1, x_2, \ldots, x_n$ and functions $f_1(x_1), f_2(x_2), \ldots, f_n(x_n)$ of each variables separately.

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It follows each function is a constant: $f_i(x_i) = \lambda_i$; these are called *separation constants*.

Example: the wave equation

$$u_{tt} = c^2 u_{xx}, \quad u(0,t) = 0 = u(L,t), \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x)$$

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- Inserting $u(x,t) = X(x)T(t)$ into homogeneous boundary conditions

$$X(0) = 0 = X(L).$$

- Get two ODEs:

$$X'' + \lambda X = 0, \quad X(0) = 0 = X(L),$$

$$T'' + c^2 \lambda T = 0.$$

Example: the wave equation cont.

Solution of eigenvalue problem:

$$X_n = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots$$

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Therefore all possible separated solutions are

$$\sin(cn\pi t/L)\sin(n\pi x/L), \quad \cos(cn\pi t/L)\sin(n\pi x/L), \quad n = 1, 2, 3, \ldots.$$

Example: the wave equation cont.

Superposition of separated solutions:

$$u(x,t) = \sum_{n=1}^{\infty} [A_n \cos(n\pi ct/L) + B_n \sin(n\pi ct/L)] \sin\left(\frac{n\pi x}{L}\right).$$

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Invoking the initial conditions

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right), \quad \psi(x) = \sum_{n=1}^{\infty} \left(\frac{n\pi c}{L}\right) B_n \sin\left(\frac{n\pi x}{L}\right).$$

These are orthogonal expansions, so coefficients are found by taking inner products with each eigenfunctions $X_n = \sin(n\pi x/L)$:

$$A_n = \frac{\langle \phi, X_n \rangle}{\langle X_n, X_n \rangle}, \quad B_n = \left(\frac{L}{n\pi c}\right) \frac{\langle \psi, X_n \rangle}{\langle X_n, X_n \rangle}.$$

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- Solution is composed of standing waves

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- These special frequencies form basis for sound waves, atomic spectra, elastic vibrations, etc.
- Notice longer strings have smaller frequencies.

Example 2: the diffusion equation

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- Satisfy the inhomogeneous boundary conditions by setting $y = 0$ and $y = H$,

$$h(x) = \sum_{n=1}^{\infty} (A_n + B_n) \sin\left(\frac{n\pi x}{L}\right)$$

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- Taking inner products with the eigenfunctions $X_n = \sin(n\pi x/L)$, one gets a system of two equations for each pair A_n, B_n

$$A_n + B_n = \frac{\langle h, X_n \rangle}{\langle X_n, X_n \rangle}, \quad A_n \exp(n\pi H/L) + B_n \exp(-n\pi H/L) = \frac{\langle g, X_n \rangle}{\langle X_n, X_n \rangle}$$