Laplace's equation in polar coordinates

Boundary value problem for disk:

$$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 0, \quad u(a,\theta) = h(\theta).$$

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- Separating variables $u = R(r)\Theta(\theta)$ gives $R''\Theta + r^{-1}R'\Theta + r^{-2}R\Theta'' = 0$ or

$$\frac{\Theta''}{\Theta} = \frac{-r^2R'' - rR'}{R} = -\lambda.$$

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- Periodic boundary conditions give rise to Fourier series with both sines and cosines as eigenfunctions.

Laplace's equation in polar coordinates, cont.

- Eigenfunctions ("circular harmonics")

$$\Theta = \begin{cases} 1 & \lambda = 0 \\ \cos(n\theta) & \lambda = n^2 \\ \sin(n\theta) & \lambda = n^2, \end{cases}$$

where $n = 1, 2, 3, \ldots$.

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- Equation for radial component is Euler equation
  $r^2 R'' + rR' - \lambda R = 0$.

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- Solutions are just powers $R = r^\alpha$; plugging in, $[\alpha(\alpha - 1) + \alpha - \lambda]r^\alpha = 0$ or $\alpha = \pm\sqrt{\lambda}$.

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- If $\lambda = 0$, get linearly independent solutions 1 and $\ln r$.
- Reject (for now) solutions involving $\ln r$ and $r^{-\alpha}$.

Laplace's equation in polar coordinates, cont.

- Superposition of separated solutions:

$$u = A_0/2 + \sum_{n=1}^{\infty} r^n [A_n \cos(n\theta) + B_n \sin(n\theta)].$$

Laplace's equation in polar coordinates, cont.

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- Satisfy boundary condition at $r = a$,

$$h(\theta) = A_0/2 + \sum_{n=1}^{\infty} a^n [A_n \cos(n\theta) + B_n \sin(n\theta)].$$

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- This is a Fourier series with cosine coefficients $a^n A_n$ and sine coefficients $a^n B_n$, so that (using the known formulas)

$$A_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \cos(n\phi) d\phi, \quad B_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \sin(n\phi) d\phi.$$

Poisson formula

Inserting the Fourier coefficient formulas into the general solution,

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)d\phi$$
$$+ \sum_{n=1}^{\infty} \frac{r^n}{\pi a^n}\int_0^{2\pi} h(\phi)[\cos(n\phi)\cos(n\theta) + \sin(n\phi)\sin(n\theta)]d\phi.$$

Poisson formula

Inserting the Fourier coefficient formulas into the general solution,

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)d\phi + \sum_{n=1}^{\infty}\frac{r^n}{\pi a^n}\int_0^{2\pi} h(\phi)[\cos(n\phi)\cos(n\theta)+\sin(n\phi)\sin(n\theta)]d\phi.$$

Use the identity
$\cos(n\phi)\cos(n\theta)+\sin(n\phi)\sin(n\theta) = \cos(n(\theta-\phi))$, and reverse the order of summation and integration

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\Big\{1 + 2\sum_{n=1}^{\infty}\Big(\frac{r}{a}\Big)^n \cos(n(\theta-\phi))\Big\}d\phi.$$

Poisson formula, cont.

Sum is geometric series in disguise:

$$\begin{aligned}
1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos(n(\theta - \phi)) &= 1 + 2\mathrm{Re}\sum_{n=1}^{\infty} \left(\frac{re^{i(\theta-\phi)}}{a}\right)^n \\
&= 1 + 2\mathrm{Re}\frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}} \\
&= \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \phi) + r^2}.
\end{aligned}$$

Poisson formula, cont.

Sum is geometric series in disguise:

$$\begin{aligned} 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n(\theta-\phi)) &= 1 + 2\mathrm{Re}\sum_{n=1}^{\infty}\left(\frac{re^{i(\theta-\phi)}}{a}\right)^n \\ &= 1 + 2\mathrm{Re}\frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}} \\ &= \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\phi) + r^2}. \end{aligned}$$

This results in Poisson's formula:

$$u(r,\theta) = \int_0^{2\pi} P(r,\theta-\phi)h(\phi)d\phi, \quad P(r,\theta) = \frac{1}{2\pi}\frac{a^2 - r^2}{a^2 - 2ar\cos(\theta) + r^2}.$$

Consequences of the Poisson formula

At $r = 0$, notice the integral is easy to compute:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)d\phi, \quad = \frac{1}{2\pi}\int_0^{2\pi} u(a,\phi)d\phi.$$

- Therefore if $\Delta u = 0$, the value of u at any point is just the average values of u on a circle centered on that point. (“Mean value theorem")
- The maximum and minimum values of u are therefore always on the domain boundary (this is true for any shape domain).