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Strategy:

1. Find a particular solution. This is often just an educated guess.
2. Formulate homogeneous problem for $w = u - u_p$ by subtracting the equations and ALL side conditions satisfied by both u and u_p.
3. Now problem for w is suitable for separation of variables.

Inhomogeneous equation, example 1

Example: diffusion equation with nonzero boundary conditions

$$u_t = Du_{xx}, \quad u(0,t) = u_l, \quad u(L,t) = u_r, \quad u(x,0) = \phi(x).$$

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Guess $u_p = Ax + B$. To get $u_p(0,t) = u_l$ and $u_p(L,t) = u_r$, need $B = u_l$ and $A = (u_r - u_l)/L$.

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Then $w = u - u_p$ solves equation with homogeneous boundary conditions (check this!)

$$w_t = Dw_{xx}, \quad w(0,t) = 0, \quad w(L,t) = 0, \quad w(x,0) = \phi(x) - u_p.$$

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Our complete solution will be $u(x,t) = w(x,t) + u_p(x)$, where from before

$$\begin{aligned} w(x,t) &= \sum_{n=1}^{\infty} A_n \exp(-D(n\pi/L)^2 t) \sin\left(\frac{n\pi x}{L}\right), \\ A_n &= \frac{\langle \phi - u_p, X_n \rangle}{\langle X_n, X_n \rangle}. \end{aligned}$$

Inhomogeneous equation, example 2

Poisson equation

$$u_{xx} + u_{yy} = 1,\ u(0, y) = 0 = u(1, y),\ u(x, 0) = 0 = u(x, 1)$$

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Complete solution is $u(x,t) = w(x,t) + u_p(x)$; using previous result,

$$w(x,y) = \sum_{n=1}^{\infty} [A_n \exp(n\pi y) + B_n \exp(-n\pi y)] \sin(n\pi x),$$

where

$$A_n + B_n = -\frac{\langle \frac{1}{2}x(x-1), X_n \rangle}{\langle X_n, X_n \rangle}, \quad A_n \exp(n\pi) + B_n \exp(-n\pi) = -\frac{\langle \frac{1}{2}x(x-1), X_n \rangle}{\langle X_n, X_n \rangle}$$

and

$$\frac{\langle \frac{1}{2}x(x-1), X_n \rangle}{\langle X_n, X_n \rangle} = -\frac{2}{\pi^4 n^3}[\cos(\pi n) - 1].$$