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Idea: find characteristic $X(T)$ and the solution on characteristic U *simultaneously*, in a way that is compatible with the initial condition.

Example

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Solving for U gives

$$U = u(x,t) = \frac{x}{1 + 3t}.$$

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1. Characteristics can intersect. This produces *shock waves*.
2. Characteristics don't necessarily pass through every point (x, t) for $t > 0$. This produces *rarefaction waves*.

Shocks

Consider

$$u_t + uu_x = 0, u(x,0) = \begin{cases} 2 & x < 0 \\ 1 & x > 0 \end{cases}$$

For $x < 0$ have speed $= 2$, for $x > 0$ speed $= 1$, and characteristics intersect in region $t < x < 2t$.

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Resolution: allow the solution to be *discontinuous*:

$$u(x,t) = \begin{cases} 2 & x < x_s(t) \\ 1 & x > x_s(t) \end{cases}$$

where $x_s(t)$ is a curve called a *shock*. But what determines x_s?

Shock speed

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Plugging into $u_t + J(u)_x = 0$,

$$-c_s U' + J(U)' = 0.$$

Integration over $\mathbb{R}$ gives $-c_s(u_R - u_L) + J(u_R) - J(u_L) = 0$ or

$$c_s = \frac{J(u_R) - J(u_L)}{u_R - u_L}, \quad \text{(Rankine-Hugoniot condition)}$$

Example, continued

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Shock evolves according to

$$dx_s/dt = [J(2) - J(1)]/(2-1) = 3/2, \quad x_s(0) = 0,$$

so that $x_s(t) = 3/2t$.

Rarefactions

Now consider

$$u_t + uu_x = 0, \quad u(x,0) = \begin{cases} 1 & x < 0 \\ 2 & x > 0 \end{cases}$$

Problem: no characteristics fill region $t < x < 2t$, called a *rarefaction*.

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Therefore complete solution is

$$u(x,t) = \begin{cases} 1 & x < t \\ x/t & t < x < 2t \\ 2 & x > 2t. \end{cases}$$

Example 1 (non-constant shock speed).

Consider

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Conservation form is $u_t + J(u)_x = 0$ with $J = u^2/2$, so

$$x_s'(t) = \frac{J(2x_s/(1+2t)) - J(0)}{2x_s/(1+2t) - 0} = \frac{x_s}{1+2t}.$$

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Solving this differential equation with initial condition $x_s(0) = 1$ gives

$$x_s(t) = \sqrt{1 + 2t}.$$

Example 2 (shock creation)

Consider

$$u_t + (u+1)u_x = 0, \quad u(x,0) = \begin{cases} 1 & x < 0 \\ 1 - x & 0 < x < 1 \\ 0 & x > 1. \end{cases}$$

Characteristics which start at $x = 0$ and $x = 1$ are lines $X_1(T) = 2T$ and $X_2(T) = 1 + T$.

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Characteristics which start at $x = 0$ and $x = 1$ are lines $X_1(T) = 2T$ and $X_2(T) = 1 + T$. These intersect at $X = 2$ and $T = 1$ to create shock, evolving as

$$x_s'(t) = \frac{J(0) - J(1)}{0 - 1}, \quad J(u) = u^2/2 + u, \quad x_c(1) = 2.$$

so that $x_s(t) = 3/2(t-1) + 2$.

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In region $X_1(t) < x < X_2(t)$, characteristics with $X(t) = x$ satisfy $X'(T) = (U+1)$.

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In region $X_1(t) < x < X_2(t)$, characteristics with $X(t) = x$ satisfy $X'(T) = (U+1)$. They are therefore lines $X(T) = (U+1)(T-t) + x$.

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so that $x_s(t) = 3/2(t-1) + 2$.

In region $X_1(t) < x < X_2(t)$, characteristics with $X(t) = x$ satisfy $X'(T) = (U+1)$. They are therefore lines $X(T) = (U+1)(T-t) + x$. To be consistent with the initial condition,

$$U = u(X(0), 0) = 1 - [x - (U+1)t],$$

so that

$$U = u(x,t) = \frac{1 - x + t}{1 - t}.$$

Notice that this solution breaks down at $t = 1$.

Example 3 (multiple shocks)

Consider

$$u_t + 3u^2 u_x = 0, \qquad u(x,0) = \begin{cases} 3 & x < 1 \\ 2 & 1 < x < 2 \\ 1 & x > 2. \end{cases}$$

For small t, three regions with characteristic speeds

$$X'(T) = 3U^2 = \begin{cases} 27 & x < x_1(t) \\ 12 & x_1(t) < x < x_2(t) \\ 3 & x > x_2(t). \end{cases}$$

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Conservation form $u_t + J(u)_x = 0$ has $J(u) = u^3$, so that

$$x_1'(t) = \frac{2^3 - 3^3}{2 - 3} = 19, \qquad x_1(0) = 1$$

$$x_2'(t) = \frac{1^3 - 2^3}{1 - 2} = 7, \qquad x_2(0) = 2$$

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Solution:

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These intersect where $t = 1/12$ and $x = 31/12$, producing a single shock with $u = 3$ on left and $u = 1$ on right.

Example 3,cont.

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Third shock $x_3(t)$ evolves according to

$$x_3'(t) = \frac{1^3 - 3^3}{1 - 3} = 13, \qquad x_3(1/12) = 32/12,$$

so that $x_3(t) = 13(t - 1/12) + 31/12$.