Symmetries of partial differential equations

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For example, if $u(x,t)$ solves $u_t = u_{xx}$, so does $u(x - x_0, t - t_0)$. This is a *translation* symmetry.

Also, if $u(x,t)$ solves $u_t = u_{xx}$, so does $u(-x, t)$. This is a *reflection* symmetry. Note $u(x, -t)$ is not a solution, however!

Symmetry transformations

Given a function $u(x,t)$, transform both independent and dependent variables by a mapping

$$(x,t,u) \to (x',t',u')$$

or explicitly

$$x' = X(x,t,u), \quad t' = T(x,t,u), \quad u' = U(x,t,u).$$

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A partial differential equation is said to have a *symmetry* if

$$u'(x, t) = U(x, t, u(X(x, t, u), T(x, t, u)))$$

is a solution, given that $u(x, t)$ is.

Example 1

Consider equation

$$u_t = uu_{xx}$$

and transformation

$$x' = -x, \quad t' = -t, \quad , u' = -u.$$

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Given solution $u(x,t)$, check $u'(x,t) = -u(-x,-t)$ also works:

$$\partial_t \Big(-u(-x,-t) \Big) = u_t(-x,-t), \quad \text{and}$$

$$\Big(-u(-x,-t) \Big) \partial_{xx} \Big(-u(-x,-t) \Big) = u(-x,-t) u_{xx}(-x,-t).$$

Example 2

Consider the nonlinear Schrödinger equation

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u = 0, \quad u(x,t) : \mathbb{R}^2 \to \mathbb{C}.$$

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$$U(x,t,u) = \exp(-iv(x + vt/2))u, \quad X = x + vt, \quad T = t.$$

Check $w(x,t) = \exp(-iv(x + Vt/2))u(x + Vt, t)$ is a solution:

$$iw_t + \frac{1}{2}w_{xx} + |w|^2 w =$$

$$e^{-iV(x+Vt/2)} \left\{ i\left(\frac{-iV^2}{2}u + u_t + Vu_x \right) + \frac{1}{2}\left(-V^2 u - 2iVu_x + u_{xx} \right) + |u|^2 u \right\}$$

$$= e^{-iV(x+Vt/2)} \left\{ u_t + \frac{1}{2}u_{xx} + |u|^2 u \right\} = 0.$$

Dilation symmetries

Special type of symmetry involves rescaling space and time

$$x' = \frac{x}{L}, \quad t' = \frac{t}{L^\beta},$$

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$$\frac{u_t(x/L, t/L^\beta)}{L^\beta} + c\frac{u_x(x/L, t/L^\beta)}{L} = 0,$$

therefore need $\beta = 1$.

For example, given solution $u = \sin(x - ct)$, can construct a new solution $u = \sin([x - ct]/L)$.

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Remedy: rewrite as

$$L^{-1} u_x(x/L, y/L^\beta) + L^{\beta} (y/L^\beta)^2 u_y(x/L, y/L^\beta) = 0.$$

which means $\beta = -1$.

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Substitute $L^{-\gamma} u(x/L, t/L^{\beta})$ for u,

$$L^{-\gamma-\beta} u_t = L^{-2\gamma-2} u u_{xx} - L^{-\gamma-1} u_x, \quad \text{evaluated at } (x/L, t/L^{\beta}).$$

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It follows $\gamma + \beta = 2\gamma + 2 = \gamma + 1$, so that $\beta = 1$ and $\gamma = -1$.

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Similarity solutions are often physically important since they are scale invariant.

Invariance of similarity solutions

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Conversely, if $u(x,t)$ is invariant under symmetry transformation, can be written in terms of new variables $w(\eta,\xi)$, where

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Differentiating with respect to L and setting $L = 1$,

$$w_\xi(\eta, \xi/L^{\beta+1})\xi = 0,$$

which means $w = w(\eta)$.

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$$Df''(\eta) + \frac{\eta}{2} f'(\eta) = 0.$$

Separation of variables gives

$$f'(\eta) = Ce^{-\eta^2/4D},$$

therefore

$$f(\eta) = C_1 \text{erf}(\eta/4D) + C_2, \quad \text{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy.$$

More general similarity solutions

For the more general dilation symmetry,

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These solutions are invariant under the transformation:

$$\begin{aligned} L^{-\gamma} u(x/L, t/L^\beta) &= L^{-\gamma} (t/L^\beta)^{-\gamma/\beta} f\left(\frac{x/L}{(t/L^\beta)^{1/\beta}}\right) \\ &= t^{-\gamma/\beta} f\left(\frac{x}{t^{1/\beta}}\right) = u(x, t). \end{aligned}$$

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Plugging into equation gives

$$-\frac{1}{2}f - \frac{\eta}{4}f' = ff'' - f^3.$$

Need to find solution numerically!

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Example: for diffusion equation $u_t = Du_{xx}$, inserting $L^{-\gamma}u(x/L, t/L^{\beta})$ into the equation gives $L^{-\gamma-\beta}u_t = DL^{-\gamma-2}u_{xx}$. Thus $\beta = 2$ but γ is arbitrary.

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Already found solution with $\gamma = 0$,

$$f(\eta) = C_1 \mathrm{erf}(\eta/4D) + C_2,$$

but this does not decay at $\pm\infty$.

Incomplete similarity, cont.

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Inserting solution of form $u = t^{\alpha} f(x/t^{1/\beta})$ and changing variables

$$\frac{d}{dt}\int_{-\infty}^{\infty} t^{\alpha+1/\beta} f(\eta)d\eta = 0,$$

therefore $\alpha = -1/\beta = -1/2$.

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Substituting $u = t^{-1/2} f(x/t^{1/2})$, get ODE

$$f'' + \frac{\eta}{2}f' + \frac{1}{2}f = f'' + \frac{1}{2}(\eta f)' = 0.$$

Using the condition $f(\pm\infty) = 0$, integration gives

$$f' = -\frac{1}{2}\eta f, \quad df/f = -\frac{1}{2}\eta d\eta, \quad f(\eta) = Ce^{-\eta^2/4}.$$

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With $C = 1/\sqrt{4\pi}$ get fundamental solution

$$u = (4\pi t)^{-1/2} e^{-x^2/4t}$$

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Either $f' = 0$, or $f = \eta$. Latter choice leads to solution

$$u(x,t) = \eta = \frac{x}{t}.$$

i.e. a rarefaction wave.

Similarity solutions,Example 2

A model for a convective thermal layer is

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Note: to make this like original equation, y should be replaced by y/L everywhere. Write as:

$$\frac{3(y/L)}{L^{\beta-1}} u_x(x/L^\beta, y/L) - \frac{1}{L^2} u_{yy}(x/L^\beta, y/L) = 0.$$

therefore $u(x/L^\beta, y/L)$ is still a solution if $\beta = 3$.

Similarity solutions,Example 2, cont.

Plug in similarity ansatz $u = f(\eta)$, $\eta = y/x^{1/3}$,

$$-\eta^2 f'(\eta) - f''(\eta) = 0, \quad f(0) = 0, \quad \lim_{\eta\to\infty} f(\eta) = 1.$$

Similarity solutions,Example 2, cont.

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$$-\eta^2 f'(\eta) - f''(\eta) = 0, \quad f(0) = 0, \quad \lim_{\eta\to\infty} f(\eta) = 1.$$

Separate variables $f''/f' = -\eta^2$ and integrate

$$f' = Ae^{-\eta^3/3},$$

and integrate again

$$f = A\int_0^\eta e^{-s^3/3} ds + B.$$

Using the boundary conditions, $B = 0$ and

$$A = \left(\int_0^\infty e^{-s^3/3} ds\right)^{-1}.$$

Similarity solutions, Example 3

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Inserting $L^{-\gamma} v(r/L, t/L^{\beta})$

$$\frac{1}{L^{\beta+\gamma}} v_t(r/L, t/L^{\beta}) = \frac{1}{L^{2+\gamma}} \left(\frac{1}{(r/L)}[(r/L)v(r/L, t/L^{\beta})]_r\right)_r,$$

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thus $\beta = 2$ but γ is undetermined.
To be compatible with initial condition, insert $L^{-\gamma} v(r/L, 0)$

$$L^{-\gamma} v(r/L, 0) = L^{-1}(r/L)^{-1}, \quad \text{therefore } \gamma = 1.$$

Similarity solutions, Example 3 cont.

Since

$$t^{-1/2} f(r/t^{1/2}) = \frac{1}{r}[\eta^{-1} f(r/t^{1/2})],$$

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In original variables

$$v(r,t) = \frac{1}{r}\left(1 + B\exp\left(-\frac{r^2}{4t}\right)\right).$$

For bounded solution at origin, $B = -1$.

Similarity solutions, Example 4.

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$$u_t = (uu_x)_x.$$

If $L^{-\gamma}u(r/L, t/L^{\beta})$ is a solution, need $\beta = 2 + \gamma$.

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Inserting

$$u = t^{-\gamma/\beta} f(\eta), \quad \eta = x/t^{1/\beta},$$

then

$$\int_{-\infty}^{\infty} u(x,t)dx = t^{(1-\gamma)/\beta}\int_{-\infty}^{\infty} f(\eta)d\eta$$

which means that $\gamma = 1$, $\beta = 3$.

Similarity solutions, Example 4, cont.

Similarity solution $u = t^{-1/3} f(\eta)$, $\eta = x/t^{1/3}$ solves

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Solution: just take positive part, so that

$$u(x,t) = \begin{cases} t^{-1/3}\left(B - \frac{x^2}{6t^{2/3}}\right) & x^2 < 6Bt^{2/3}, \\ 0 & x^2 > 6Bt^{2/3}. \end{cases}$$

Observation: value of B determines total mass $\int u dx$, and this is constant in time.