First order wave equations

Transport equation is conservation law with $J = cu$,

$$u_t + cu_x = 0, \quad -\infty < x < \infty.$$

First order wave equations

Transport equation is conservation law with $J = cu$,

$$u_t + cu_x = 0, \quad -\infty < x < \infty.$$

Regard as directional derivative $\nabla u \cdot \mathbf{v} = (u_x, u_t) \cdot \mathbf{v} = 0$, with $\mathbf{v} = (c, 1)$.

First order wave equations

Transport equation is conservation law with $J = cu$,

$$u_t + cu_x = 0, \quad -\infty < x < \infty.$$

Regard as directional derivative $\nabla u \cdot \mathbf{v} = (u_x, u_t) \cdot \mathbf{v} = 0$, with $\mathbf{v} = (c, 1)$.

Therefore u is a constant on lines parallel to $\mathbf{v}$, having form $x - ct = x_0$. These are known as *characteristic curves*.

First order wave equations

Transport equation is conservation law with $J = cu$,

$$u_t + cu_x = 0, \quad -\infty < x < \infty.$$

Regard as directional derivative $\nabla u \cdot \mathbf{v} = (u_x, u_t) \cdot \mathbf{v} = 0$, with $\mathbf{v} = (c, 1)$.

Therefore u is a constant on lines parallel to $\mathbf{v}$, having form $x - ct = x_0$. These are known as *characteristic curves*.

It follows u is function of $x - ct$ alone:

$$u(x, t) = f(x - ct).$$

First order wave equations

More generally

$$au_x + bu_y = 0$$

has characteristic curves f the form $bx - ay = C$, and general solution is therefore

$$u(x,t) = f(bx - ay).$$

First order wave equations

More generally

$$au_x + bu_y = 0$$

has characteristic curves f the form $bx - ay = C$, and general solution is therefore

$$u(x, t) = f(bx - ay).$$

Function $f()$ is determined by boundary conditions.

More generally

$$au_x + bu_y = 0$$

has characteristic curves f the form $bx - ay = C$, and general solution is therefore

$$u(x,t) = f(bx - ay).$$

Function $f()$ is determined by boundary conditions.

Remark: since $u(x,y)$ is constant along characteristic curves, boundary data must be compatible.

First order wave equations, example

Solve $u_x = u_y$ subject to $u(x, y) = \sin(x)$ on the line $y = x$.

First order wave equations, example

Solve $u_x = u_y$ subject to $u(x, y) = \sin(x)$ on the line $y = x$.

Most general solution is $u(x, y) = f(x + y)$.

First order wave equations, example

Solve $u_x = u_y$ subject to $u(x, y) = \sin(x)$ on the line $y = x$.

Most general solution is $u(x, y) = f(x + y)$.

On the line $y = x$ we have $\sin(x) = u(x, y) = f(2x)$.

First order wave equations, example

Solve $u_x = u_y$ subject to $u(x, y) = \sin(x)$ on the line $y = x$.

Most general solution is $u(x, y) = f(x + y)$.

On the line $y = x$ we have $\sin(x) = u(x, y) = f(2x)$.

To find function $f()$, set $z = 2x$, so that $f(z) = \sin(z/2)$.

First order wave equations, example

Solve $u_x = u_y$ subject to $u(x, y) = \sin(x)$ on the line $y = x$.

Most general solution is $u(x, y) = f(x + y)$.

On the line $y = x$ we have $\sin(x) = u(x, y) = f(2x)$.

To find function $f()$, set $z = 2x$, so that $f(z) = \sin(z/2)$.

Complete solution is therefore $u(x, y) = \sin((x + y)/2)$.

Second order wave equations

Consider the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in (-\infty, \infty).$$

Second order wave equations

Consider the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in (-\infty, \infty).$$

Factor the linear operator:

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0.$$

Second order wave equations

Consider the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in (-\infty, \infty).$$

Factor the linear operator:

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0.$$

Setting $v = u_t + cu_x$, get two equations

$$\begin{aligned} v_t - cv_x &= 0, \\ u_t + cu_x &= v. \end{aligned}$$

Second order wave equations

Consider the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in (-\infty, \infty).$$

Factor the linear operator:

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0.$$

Setting $v = u_t + cu_x$, get two equations

$$\begin{aligned} v_t - cv_x &= 0, \\ u_t + cu_x &= v. \end{aligned}$$

General solution to the first equation is $v = h(x + ct)$, and

$$u_t + cu_x = h(x + ct).$$

Second order wave equations

Consider the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in (-\infty, \infty).$$

Factor the linear operator:

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0.$$

Setting $v = u_t + cu_x$, get two equations

$$\begin{aligned} v_t - cv_x &= 0, \\ u_t + cu_x &= v. \end{aligned}$$

General solution to the first equation is $v = h(x + ct)$, and

$$u_t + cu_x = h(x + ct).$$

Can show particular solution has form $u_p = f(x + ct)$;
complimentary solution has form $u_{\text{hom}} = g(x - ct)$, therefore

$$u = u_{\text{hom}} + u_p = g(x - ct) + f(x + ct).$$

Second order wave equations,cont.

Now satisfy the initial conditions:

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x)$$

Second order wave equations,cont.

Now satisfy the initial conditions:

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x)$$

Inserting into general solution

$$\begin{aligned} f(x) + g(x) &= u_0(x), \\ f'(x) - g'(x) &= \frac{1}{c} v_0(x). \end{aligned}$$

Second order wave equations,cont.

Now satisfy the initial conditions:

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x)$$

Inserting into general solution

$$\begin{aligned} f(x) + g(x) &= u_0(x), \\ f'(x) - g'(x) &= \tfrac{1}{c} v_0(x). \end{aligned}$$

Integrating the second,

$$f(x) - g(x) = \frac{1}{c} \int_0^x v_0(x') dx' + K.$$

Second order wave equations,cont.

Now satisfy the initial conditions:

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x)$$

Inserting into general solution

$$\begin{aligned} f(x) + g(x) &= u_0(x), \\ f'(x) - g'(x) &= \tfrac{1}{c} v_0(x). \end{aligned}$$

Integrating the second,

$$f(x) - g(x) = \frac{1}{c}\int_0^x v_0(x')dx' + K.$$

Use method of elimination

$$\begin{aligned} f(x) &= \tfrac{1}{2}\left(u_0(x) + \tfrac{1}{c}\int_0^x v_0(x')dx' + K\right), \\ g(x) &= \tfrac{1}{2}\left(u_0(x) - \tfrac{1}{c}\int_0^x v_0(x')dx' - K\right) \end{aligned}$$

Second order wave equations,cont.

Now satisfy the initial conditions:

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x)$$

Inserting into general solution

$$\begin{aligned} f(x) + g(x) &= u_0(x), \\ f'(x) - g'(x) &= \tfrac{1}{c} v_0(x). \end{aligned}$$

Integrating the second,

$$f(x) - g(x) = \frac{1}{c}\int_0^x v_0(x')dx' + K.$$

Use method of elimination

$$\begin{aligned} f(x) &= \tfrac{1}{2}\left(u_0(x) + \tfrac{1}{c}\int_0^x v_0(x')dx' + K\right), \\ g(x) &= \tfrac{1}{2}\left(u_0(x) - \tfrac{1}{c}\int_0^x v_0(x')dx' - K\right) \end{aligned}$$

Complete solution is therefore d'Alembert's formula

$$u = f(x+ct)+g(x-ct) = \frac{1}{2}\left(u_0(x+ct) + u_0(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(x')dx'.$$

Generalized second order equation

Notice that

$$u_{xx} + (a-b)u_{xy} - abu_{yy} = 0$$

can be factored as

$$\left(\frac{\partial}{\partial x} + a\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}\right)u = 0.$$

which can be written as the system

$$\begin{aligned} v_x + av_y &= 0, \\ u_x - bu_y &= v. \end{aligned}$$

Generalized second order equation

Notice that

$$u_{xx} + (a-b)u_{xy} - abu_{yy} = 0$$

can be factored as

$$\left(\frac{\partial}{\partial x} + a\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}\right)u = 0.$$

which can be written as the system

$$\begin{aligned} v_x + av_y &= 0, \\ u_x - bu_y &= v. \end{aligned}$$

Using same process as before, general solution is

$$u(x,y) = f(y-ax) + g(y+bx)$$

Generalized second order equation, example

Want a d'Alembert-type solution for $u_{xx} + u_{xy} - 20u_{yy} = 0$ subject to initial conditions $u(x, 0) = \phi(x)$ and $u_y(x, 0) = \psi(x)$.

Generalized second order equation, example

Want a d'Alembert-type solution for $u_{xx} + u_{xy} - 20u_{yy} = 0$ subject to initial conditions $u(x,0) = \phi(x)$ and $u_y(x,0) = \psi(x)$. Factoring gives

$$\left(\frac{\partial}{\partial x} + 5\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial y}\right) u = 0.$$

so that general solution is

$$u(x,y) = g(4x + y) + f(5x - y).$$

Generalized second order equation, example

With general solution $u(x,t) = g(4x+y) + f(5x-y)$, satisfying initial data gives

$$g(4x) + f(5x) = \phi(x),$$
$$g'(4x) - f'(5x) = \psi(x).$$

Generalized second order equation, example

With general solution $u(x,t) = g(4x+y) + f(5x-y)$, satisfying initial data gives

$$g(4x) + f(5x) = \phi(x),$$
$$g'(4x) - f'(5x) = \psi(x).$$

Integration of the second equation gives

$$\frac{1}{4}g(4x) - \frac{1}{5}f(5x) = \int_0^x \psi(x')dx' + C.$$

Generalized second order equation, example

With general solution $u(x,t) = g(4x+y) + f(5x-y)$, satisfying initial data gives

$$\begin{aligned} g(4x) + f(5x) &= \phi(x), \\ g'(4x) - f'(5x) &= \psi(x). \end{aligned}$$

Integration of the second equation gives

$$\frac{1}{4}g(4x) - \frac{1}{5}f(5x) = \int_0^x \psi(x')dx' + C.$$

Using elimination to solve,

$$\begin{aligned} g(4x) &= \frac{4}{9}\phi(x) + \frac{20}{9}\int_0^x \psi(x')dx' + \frac{20C}{9}, \\ f(5x) &= \frac{5}{9}\phi(x) - \frac{20}{9}\int_0^x \psi(x')dx' - \frac{20C}{9}. \end{aligned}$$

Generalized second order equation, example

With general solution $u(x,t) = g(4x+y) + f(5x-y)$, satisfying initial data gives

$$g(4x) + f(5x) = \phi(x),$$
$$g'(4x) - f'(5x) = \psi(x).$$

Integration of the second equation gives

$$\frac{1}{4}g(4x) - \frac{1}{5}f(5x) = \int_0^x \psi(x')dx' + C.$$

Using elimination to solve,

$$g(4x) = \frac{4}{9}\phi(x) + \frac{20}{9}\int_0^x \psi(x')dx' + \frac{20C}{9},$$
$$f(5x) = \frac{5}{9}\phi(x) - \frac{20}{9}\int_0^x \psi(x')dx' - \frac{20C}{9}.$$

It follows that

$$u(x,y) = g(4x+y) + f(5x-y) = g(4(x+y/4)) + f(5(x-y/5))$$
$$= \frac{4}{9}\phi(x+y/4) + \frac{4}{9}\phi(x-y/5) + \frac{20}{9}\int_{x-y/5}^{x+y/4} \psi(x')dx'.$$